Dual-family viscous shock waves in n conservation laws with application to multi-phase flow in porous media*

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Abstract

We consider shock waves satisfying the viscous profile criterion in general systems of n conservation laws. We study $S_{i,j}$ dual-family shock waves, which are associated with a pair of characteristic families i and j. We explicitly introduce defining equations relating states and speeds of $S_{i,j}$ shocks, which include the Rankine–Hugoniot conditions and additional equations resulting from the viscous profile requirement. Then we develop a constructive method for finding the general local solution of the defining equations for such shocks and derive formulae for the sensitivity analysis of $S_{i,j}$ shocks under change of problem parameters. All possible structures of solutions of the Riemann problems containing $S_{i,j}$ shocks and classical waves are described. As a physical application, all types of $S_{i,j}$ shocks with $i > j$ are detected and studied in a family of models for multi-phase flow in porous media.

Keywords: Dual-family shock, viscous profile, conservation laws, sensitivity analysis, Riemann problem, multi-phase flow, porous medium

1 Introduction

In this paper, shock waves in general systems of n conservation laws in one space dimension x are considered. When shock waves are required to possess viscous profiles rather than to satisfy Lax's inequalities, new types of shocks arise. In general, these

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shocks may be associated with the i-th characteristic family on the left and the j-th characteristic family on the right. We call such waves $S_{i,j}$ dual-family shocks. It was shown in [5] (see also [6]) that for $i > j$ the viscous profile requirement provides exactly the number of additional equations ($i - j$ equations) that is necessary to ensure that the number of characteristics emanating from the shock in positive time direction equals the number of independent equations at the shock interface.

For systems of two equations, transitional shock waves ($i = j + 1$) were studied in [4, 12, 14], and novel structures of Riemann solutions resulting from such shocks were described. Shock waves with one or several additional equations for the viscous profile were found in problems of wave propagation in ferromagnetics, composite elastic media, elastic beams, and MHD, see [6], and in three phase flow in porous media they were analyzed for the case $S_{2,1}$ in [11]. A program for studying stability in the sense of Hadamard of $S_{i,j}$ shocks was presented in [7], where a simple example of $S_{3,1}$ shock was exhibited; another example of $S_{3,1}$ shock can be found in [8].

In this paper, we explicitly introduce defining equations that relate states and speeds of $S_{i,j}$ shocks. These equations include Rankine–Hugoniot conditions (basic equations) and additional equations resulting from the viscous profile requirement. Then we develop a constructive method for perturbation analysis of general dual-family shocks under parameter change, in which relationships between states at opposite sides of the shock and shock speed resulting from perturbations of problem parameters are derived.

The role of $S_{i,j}$ shocks in solutions of the Riemann problem is described. It turns out that $S_{i,j}$ shocks with $i > j$ may appear in generic Riemann solutions. The presence of $S_{i,j}$ shocks with $i > j + 1$ leads to the repetition of separated classical waves of the same characteristic family in a Riemann solution.

As a physical application, we consider a flow of multi-phase fluid through porous medium with the quadratic Corey model for relative permeabilities of fluid phases. For the identity viscosity matrix, we find analytically $S_{i,j}$ shocks for any $i > j$. The method for finding these shocks by continuation for state-dependent viscosity matrices of Corey type models is discussed. The variety and form of dual-family shocks may be important for applications such as oil and gas recovery.

The paper is organized as follows. Dual-family shocks are introduced in Section 2. Section 3 contains qualitative study of viscous profile requirement and defining equations. The defining equations relating states and speeds of $S_{i,j}$ shocks are explicitly given in Section 4, and variation of these equations under change of problem parameters is studied. Section 5 discusses structures of Riemann solutions containing $S_{i,j}$ shocks. In Section 6, $S_{i,j}$ shocks are analyzed in multi-phase flow through porous medium. The conclusion summarizes the contribution. Some technical proofs are given in the Appendix.

2 Dual-family shock waves

We consider systems of partial differential equations of the form

$$\frac{\partial G(U)}{\partial t} + \frac{\partial F(U)}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(D(U) \frac{\partial U}{\partial x} \right), \quad t \geq 0, \ x \in \mathbb{R} \tag{2.1}$$

in the vanishing viscosity limit $\varepsilon \searrow 0$. The function representing conserved quantities $G(U) \in \mathbb{R}^n$, the flux function $F(U) \in \mathbb{R}^n$, and the $n \times n$ viscosity matrix $D(U)$ depend smoothly on the state vector $U \in \mathbb{R}^n$. Taking $\varepsilon = 0$ in (2.1) yields a system of n first-order conservation laws

$$\frac{\partial G(U)}{\partial t} + \frac{\partial F(U)}{\partial x} = 0. \tag{2.2}$$

Real eigenvalues $\lambda(U)$ of the characteristic equation $\det(\partial F/\partial U - \lambda\, \partial G/\partial U) = 0$ are the characteristic speeds. Assuming that all the eigenvalues are real and distinct in a region of state space U (the strictly hyperbolic region), we list them in increasing order $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.

A shock wave is a discontinuous (weak) solution of system (2.2) consisting of a left state $U_- = \lim_{x/t \nearrow s} U(x,t)$ and a right state $U_+ = \lim_{x/t \searrow s} U(x,t)$, where s is the shock speed. A shock wave is considered admissible if there is a traveling wave solution (or viscous profile) $U(x,t) = U(\zeta)$, $\zeta = (x - st)/\varepsilon$ of system (2.1), which represents the shock in the vanishing viscosity limit $\varepsilon \searrow 0$. Substituting this solution into (2.1) and integrating over ζ, we find that $U(\zeta)$ is a solution (orbit) of the system of ordinary differential equations

$$D(U)\dot{U} = F(U) - F(U_-) - s(G(U) - G(U_-)), \tag{2.3}$$

"connecting" the left equilibrium $U(-\infty) = U_-$ to the right equilibrium $U(+\infty) = U_+$; the dot denotes the derivative with respect to ζ.

By linearizing equation (2.3) about the equilibria U_- and U_+ we obtain

$$\Delta\dot{U} = B(U_\pm, s)\Delta U, \qquad \Delta U(\zeta) = U(\zeta) - U_\pm, \tag{2.4}$$

where $B(U,s)$ is the $n \times n$ matrix

$$B(U,s) = \frac{\partial}{\partial U}\Big[D^{-1}(U)\big(F(U) - F(U_-) - s(G(U) - G(U_-))\big)\Big]. \tag{2.5}$$

Let $\mu_i(U,s)$, $i = 1,\dots,n$ be the eigenvalues of the matrix $B(U,s)$ ordered with increasing real parts $\operatorname{Re}\mu_1 \le \operatorname{Re}\mu_2 \le \cdots \le \operatorname{Re}\mu_n$.

Let us define an $S_{i,j}$ shock as a shock possessing a viscous profile and satisfying the inequalities

$$S_{i,j}: \quad \begin{array}{l} \operatorname{Re}\mu_{i-1}(U_-,s) < 0 < \operatorname{Re}\mu_i(U_-,s), \\ \operatorname{Re}\mu_j(U_+,s) < 0 < \operatorname{Re}\mu_{j+1}(U_+,s) \end{array} \tag{2.6}$$

(if $i-1 = 0$ or $j+1 = n+1$, the corresponding inequality is disregarded). It is easy to see that $\mu_i(U_-,s) = 0$ and $\mu_j(U_+,s) = 0$ if $s = \lambda_i(U_-)$ and $s = \lambda_j(U_+)$, respectively. Under rather general conditions (see e.g. [6, 10]), inequalities (2.6) reduce to

$$S_{i,j}: \quad \begin{array}{l} \lambda_{i-1}(U_-) < s < \lambda_i(U_-), \\ \lambda_j(U_+) < s < \lambda_{j+1}(U_+). \end{array} \tag{2.7}$$

For $i = j$, inequalities (2.7) are the Lax conditions. Thus, an $S_{i,i}$ shock is a classical i-shock. Shocks with $i < j$ are called overcompressive. For $i = j+1$ such a shock is termed transitional or undercompressive. The inequalities in the first row of (2.7) coincide with the Lax conditions for the left state U_- of an i-shock. Analogously, the inequalities in the second row of (2.7) coincide with the Lax conditions for the right state U_+ of a j-shock. Therefore, the $S_{i,j}$ shock can be seen as a *dual-family shock wave* associated with the i-th characteristic family on the left and with the j-th characteristic family on the right.

For characteristic shocks, the shock speed coincides with a characteristic speed at left, right, or both sides. As it was noticed above, the eigenvalues $\mu_i(U_-, s)$ or/and $\mu_j(U_+, s)$ vanish in these cases. Thus, we can distinguish three types of characteristic shocks as

$$\begin{array}{ll} S_{i,j}^- : & \mu_i(U_-, s) = 0, \quad \operatorname{Re}\mu_j(U_+, s) < 0 < \operatorname{Re}\mu_{j+1}(U_+, s); \\ S_{i,j}^+ : & \operatorname{Re}\mu_{i-1}(U_-, s) < 0 < \operatorname{Re}\mu_i(U_-, s), \quad \mu_j(U_+, s) = 0; \\ S_{i,j}^\pm : & \mu_i(U_-, s) = \mu_j(U_+, s) = 0. \end{array} \tag{2.8}$$

Conditions (2.8) can be written in terms of characteristic speeds as

$$\begin{array}{ll} S_{i,j}^- : & s = \lambda_i(U_-), \quad \lambda_j(U_+) < s < \lambda_{j+1}(U_+); \\ S_{i,j}^+ : & \lambda_{i-1}(U_-) < s < \lambda_i(U_-), \quad s = \lambda_j(U_+); \\ S_{i,j}^\pm : & s = \lambda_i(U_-) = \lambda_j(U_+). \end{array} \tag{2.9}$$

3 Defining equations

Let us consider $S_{i,j}$ as a point in the space (U_-, U_+, s) of the left and right states and speed of shocks. For each i, j, the set of all $S_{i,j}$ shocks can be expected to define a smooth surface $\mathcal{S}_{i,j}$ in the space (U_-, U_+, s); see [3] for an example of such a surface for transitional shocks and quadratic flow functions. Locally this surface can be given by a maximal rank system of equations. Following [6, 12], we distinguish the basic equations defined by the system of conservation laws (2.2) and additional equations determined by the viscous profile requirement. There are n basic equations, which are the Rankine–Hugoniot conditions

$$\mathcal{H}(U_-, U_+, s) \equiv F(U_+) - F(U_-) - s(G(U_+) - G(U_-)) = 0 \in \mathbb{R}^n. \tag{3.1}$$

These equations are obtained from the condition that the shock is a weak solution of (2.2); they also follow from requiring that U_+ is an equilibrium of (2.3).

Additional n^{add} equations are denoted by

$$\mathcal{H}^{add}(U_-, U_+, s) = 0 \in \mathbb{R}^{n^{add}}. \tag{3.2}$$

The number of additional equations n^{add} can be determined by considering the intersection of the unstable manifold $\mathcal{M}_u(U_-)$ and the stable manifold $\mathcal{M}_s(U_+)$ of the equilibria $U_\pm$. The viscous profile exists if the intersection of these manifolds is not empty, see Figure 3.1(a). Using inequalities (2.6), we find

$$\dim \mathcal{M}_u(U_-) = n - i + 1, \quad \dim \mathcal{M}_s(U_+) = j. \tag{3.3}$$

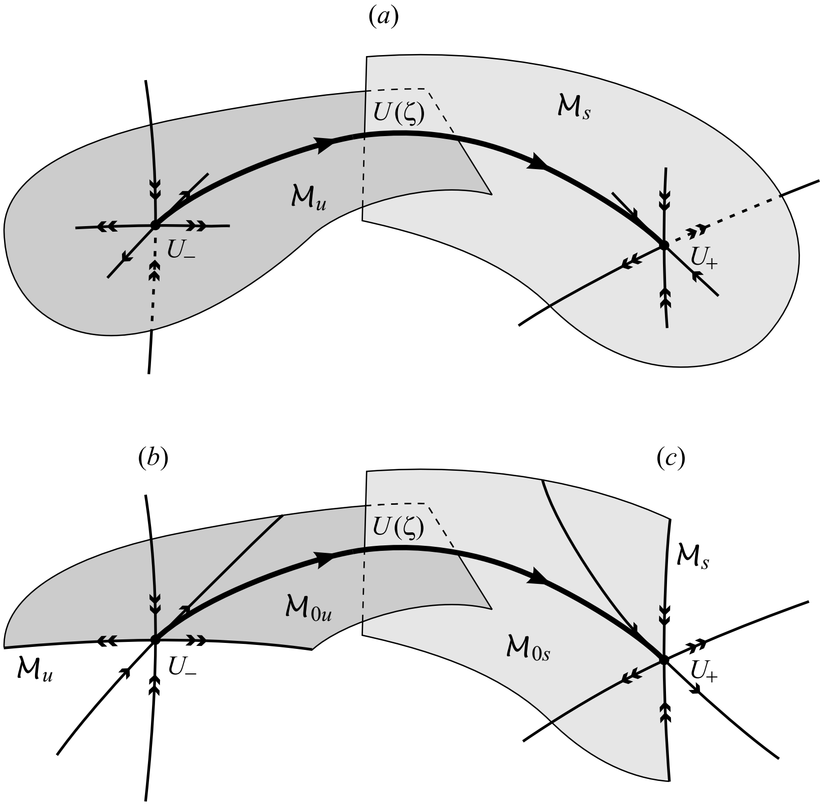

Figure 3.1: A viscous profile: (a) for $S_{i,j}$ shock, (b) and (c) for $S^{\pm}_{i,j}$ shock.

Generically, the manifolds $\mathcal{M}_u(U_-)$ and $\mathcal{M}_s(U_+)$ intersect forming a locally structurally stable phase state configuration if $\dim \mathcal{M}_u(U_-) + \dim \mathcal{M}_s(U_+) > n$, which implies $i \leq j$. In this case $n^{add} = 0$, i.e., there are no additional equations. Since the intersection of the manifolds has dimension $\dim \mathcal{M}_u(U_-) + \dim \mathcal{M}_s(U_+) - n = i - j + 1$, generically there exists a single viscous profile for classical shocks $(i = j)$ and an infinite number of viscous profiles for overcompressive shocks $(i < j)$.

In case $i > j$, the manifolds $\mathcal{M}_u(U_-)$ and $\mathcal{M}_s(U_+)$ do not intersect in general. More precisely, if $\mathcal{M}_u(U_-)$ and $\mathcal{M}_s(U_+)$ intersect and the intersection is a single orbit, then this is a singular situation (the so-called nontransversal intersection) of codimension $i - j$ [2]. The least degenerate case in this situation occurs when the tangent spaces of $\mathcal{M}_u(U_-)$ and $\mathcal{M}_s(U_+)$ have one-dimensional intersection at each point of the connecting orbit (so-called quasi-transverse intersection). Therefore, the number of additional equations (3.2) for $S_{i,j}$ shocks with $i > j$ equals $n^{add} = i - j$.

Similar surfaces $\mathcal{S}^-_{i,j}$, $\mathcal{S}^+_{i,j}$, and $\mathcal{S}^{\pm}_{i,j}$, under some genericity assumptions for the nature of nonhyperbolic equilibria, can be defined for characteristic dual-family shocks. In these cases, there are one or two equations resulting from the conditions $s = \lambda(U_-)$ or/and $s = \lambda(U_+)$. These equations are related to system (2.2) only, so that they supplement the basic equations (3.1). In case $s = \lambda_i(U_-)$, the orbits of equation (2.3) starting at U_- as $\zeta \to -\infty$ generally form a smooth manifold $\mathcal{M}_{0u}(U_-)$ of dimension $n - i + 1$ [16]. This manifold has a boundary, which is the unstable manifold $\mathcal{M}_u(U_-)$

of dimension $n - i$, see Figure 3.1(b). We consider the generic situation when the viscous profile $U(\zeta)$ does not lie in the boundary. Similarly, in case $s = \lambda_j(U_+)$, orbits finishing at U_+ as $\zeta \to +\infty$ form a smooth manifold $\mathcal{M}_{0s}(U_+)$ of dimension j. This manifold has a boundary, which is the stable manifold $\mathcal{M}_s(U_+)$ of dimension $j-1$, see Figure 3.1(c). We assume that the viscous profile $U(\zeta)$ does not lie in $\mathcal{M}_s(U_+)$. The dimensions of $\mathcal{M}_{0u}(U_-)$ and $\mathcal{M}_{0s}(U_+)$ coincide with the dimensions (3.3) for $S_{i,j}$ shocks. Hence, the number of additional equations remains the same, $n^{add} = i - j$.

The total number of defining equations provides codimensions of the shock surfaces:

$$\begin{aligned} i \le j: \quad & \operatorname{codim} \mathcal{S}_{i,j} = n,\ \operatorname{codim} \mathcal{S}_{i,j}^- = \operatorname{codim} \mathcal{S}_{i,j}^+ = n+1, \\ & \operatorname{codim} \mathcal{S}_{i,j}^{\pm} = n+2; \end{aligned} \tag{3.4}$$

$$\begin{aligned} i > j: \quad & \operatorname{codim} \mathcal{S}_{i,j} = n+i-j,\ \operatorname{codim} \mathcal{S}_{i,j}^- = \operatorname{codim} \mathcal{S}_{i,j}^+ = n+i-j+1, \\ & \operatorname{codim} \mathcal{S}_{i,j}^{\pm} = n+i-j+2. \end{aligned} \tag{3.5}$$

We wee that the viscous profile admissibility criterion for $S_{i,j}$ shocks plays qualitatively different role in cases $i \le j$ and $i > j$. Indeed, for classical ($i = j$) and overcompressive ($i < j$) shocks, generically the viscous profile is a structurally stable connecting orbit that persists under small perturbations of shock states and speed satisfying the Rankine-Hugoniot conditions (3.1). However, for $S_{i,j}$ shocks with $i > j$, the existence of viscous profile implies additional relations between states and shock speeds. These relations depend on the form of viscous terms governed by the viscosity matrix $D(U)$.

Notice that surfaces corresponding to characteristic shocks form a part of the $\mathcal{S}_{i,j}$ surface boundary. The remaining part of the $\mathcal{S}_{i,j}$ boundary is related to bifurcations of the viscous profile, see [13].

Example. Let us consider an $S_{i,j}$ shock with states U_-^*, U_+^* and speed s^* in a system of three viscous conservation laws. There are $n = 3$ defining equations if $i \le j$ (classical and overcompressive shocks), which are the Rankine–Hugoniot conditions (3.1). These equations can be solved for U_+. As a result, there is a dual-family shock of the same type with an arbitrary left state U_- and speed s taken in neighborhoods of U_-^* and s^*; the uniquely determined right state U_+ is close to U_+^*, see Figure 3.2(a).

If $i = j + 1$ (transitional shocks $S_{2,1}$ and $S_{3,2}$), there is one additional equation, which generically can be solved for the speed s. Therefore, for each left state U_- in a neighborhood of U_-^* there is a dual-family shock of the same type with the uniquely determined right state U_+ and speed s, see Figure 3.2(b).

Finally, there are two additional equations for an $S_{3,1}$ shock. Solving these equations for s, the remaining equation defines a surface $\mathcal{S}_-$ in the space U_- passing through U_-^*. Hence, $S_{3,1}$ shocks exist only for left states U_- chosen on this surface. The speed and right state are given uniquely by the choice of a point $U_- \in \mathcal{S}_-$, see Figure 3.2(c).

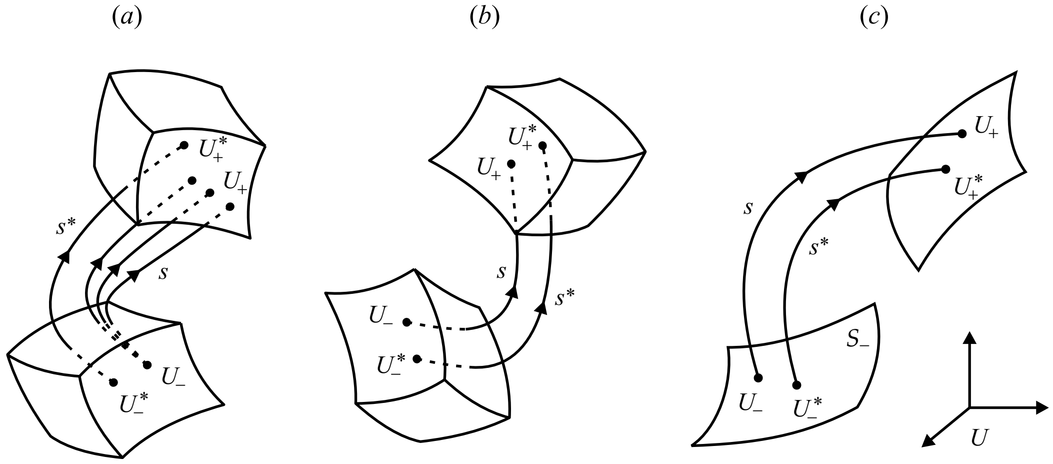

Figure 3.2: Viscous profiles of $S_{i,j}$ shocks for three conservation laws: (a) classical and overcompressive shocks, $i \leq j$, (b) $S_{2,1}$ and $S_{3,2}$ shocks (transitional shocks), (c) $S_{3,1}$ shocks.

4 Sensitivity analysis of $S_{i,j}$ shocks

In this section, we study the local form and perturbation of the shock surface $\mathcal{S}_{i,j}$. In case $i \leq j$, this information can be obtained directly by variation of the explicit Rankine–Hugoniot conditions (3.1). Thus, we will focus on the case $i > j$ when the viscous profile condition results in additional equations (3.2).

Let us determine explicitly the system of additional equations. Consider a viscous profile $U(\zeta)$ of an $S_{i,j}$ shock ($i > j$) with states $U_\pm$ and a speed s. Linearizing equation (2.3) near the solution $U(\zeta)$ yields

$$\dot{V} = B(U(\zeta), s)V, \quad V(-\infty) = V(+\infty) = 0, \quad V(\zeta) \in \mathbb{R}^n, \tag{4.1}$$

where the matrix $B(U, s)$ is given in (2.5). The corresponding adjoint linear system takes the form

$$\dot{W} = -B^T(U(\zeta), s)W, \quad W(-\infty) = W(+\infty) = 0, \quad W(\zeta) \in \mathbb{R}^n. \tag{4.2}$$

Let us denote by $\mathcal{W}$ the linear space of solutions $W(\zeta)$ of system (4.2). For any bounded function $X(\zeta) \in \mathbb{R}^n$, solutions $W(\zeta) \in \mathcal{W}$ have the property $\int_{-\infty}^{+\infty} W^T(\dot{X} - B(U(\zeta), s)X)d\zeta = 0$.

The proof of the following proposition is given in the Appendix.

Proposition 1. *For any real ζ and function $W(\zeta) \in \mathcal{W}$, the vector $W(\zeta)$ is orthogonal to both $\mathcal{M}_u(U_-)$ and $\mathcal{M}_s(U_+)$ at the point $U(\zeta)$ of the connecting orbit. In case of quasi-transverse intersection of $\mathcal{M}_u(U_-)$ and $\mathcal{M}_s(U_+)$, we have* $\dim \mathcal{W} = i - j$.

Consider a Poincaré hyperplane $\mathcal{P}$ orthogonal to the viscous profile at a fixed point $U^P = U(\zeta^P)$, see Figure 4.1. Under perturbations $U_- + \Delta U_-$, $U_+ + \Delta U_+$, and $s + \Delta s$ satisfying the Rankine-Hugoniot conditions (3.1), the stable and unstable manifolds

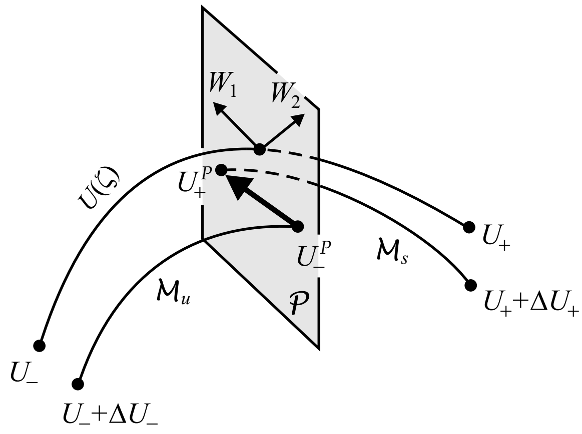

Figure 4.1: Perturbation of stable and unstable manifolds near a viscous profile.

$\mathcal{M}_u(U_- + \Delta U_-)$ and $\mathcal{M}_s(U_+ + \Delta U_+)$ change. Assuming that the perturbations are small, we can uniquely define the points $U_-^P \in \mathcal{M}_u(U_- + \Delta U_-)$ and $U_+^P \in \mathcal{M}_s(U_+ + \Delta U_+)$ such that $U_\pm^P \in \mathcal{P}$ and the vector $U_+^P - U_-^P$ is orthogonal to both $\mathcal{M}_u(U_-)$ and $\mathcal{M}_s(U_+)$ at U^P. The vector $U_+^P - U_-^P$ is a measure of the separation between the perturbed manifolds $\mathcal{M}_u(U_- + \Delta U_-)$ and $\mathcal{M}_s(U_+ + \Delta U_+)$. One can see that the manifolds $\mathcal{M}_u(U_- + \Delta U_-)$ and $\mathcal{M}_s(U_+ + \Delta U_+)$ intersect, i.e., there is a connecting orbit between $U_- + \Delta U_-$ and $U_+ + \Delta U_+$ if and only if $U_+^P = U_-^P$, see Figure 4.1.

Let us choose a basis $W_1(\zeta), \dots, W_{i-j}(\zeta)$ of $\mathcal{W}$. Then, we can introduce the function $\mathcal{H}^{add}(U_-, U_+, s)$ with values in $\mathbb{R}^{i-j}$ as

$$\begin{aligned} \mathcal{H}^{add}(U_-, U_+, s) &= \Big(W_1^T(\zeta^P)(U_+^P - U_-^P), \dots, W_{i-j}^T(\zeta^P)(U_+^P - U_-^P)\Big) \\ &= \hat{W}^T(\zeta^P)(U_+^P - U_-^P), \end{aligned} \tag{4.3}$$

where

$$\hat{W}(\zeta) = [W_1(\zeta), \dots, W_{i-j}(\zeta)] \tag{4.4}$$

is an $n \times (i-j)$ matrix. Indeed, since the vectors $W_1(\zeta^P), \dots, W_{i-j}(\zeta^P)$ are linearly independent and orthogonal to both $\mathcal{M}_u(U_-)$ and $\mathcal{M}_s(U_+)$ at U^P (see Proposition 1), the condition $U_+^P = U_-^P$ is equivalent to $\mathcal{H}^{add}(U_-, U_+, s) = 0$.

The local form of the surface $\mathcal{S}_{i,j}$ found by linearizing the defining equations near a point $(U_-, U_+, s) \in \mathcal{S}_{i,j}$ is described as follows (see the proof in the Appendix).

Theorem 1. *The tangent plane $(\Delta U_-, \Delta U_+, \Delta s)$ of the manifold $\mathcal{S}_{i,j}$ at the point $(U_-, U_+, s) \in \mathcal{S}_{i,j}$ is given by the equations $\Delta\mathcal{H} = 0$, $\Delta\mathcal{H}^{add} = 0$, where*

$$\Delta\mathcal{H} = \left(\frac{\partial F}{\partial U} - s\frac{\partial G}{\partial U}\right)_{U=U_+} \Delta U_+ - \left(\frac{\partial F}{\partial U} - s\frac{\partial G}{\partial U}\right)_{U=U_-} \Delta U_- - (G_+ - G_-)\Delta s, \tag{4.5}$$

$$\begin{aligned} \Delta\mathcal{H}^{add} = &\left(\int_{-\infty}^{+\infty} \hat{W}^T D_U^{-1} \left(\frac{\partial F}{\partial U} - s\frac{\partial G}{\partial U}\right)_{U=U_-} d\zeta\right) \Delta U_- \\ &+ \left(\int_{-\infty}^{+\infty} \hat{W}^T D_U^{-1} (G_U - G_-) d\zeta\right) \Delta s \end{aligned} \tag{4.6}$$

are the linear parts of the functions $\mathcal{H}$ and $\mathcal{H}^{add}$ evaluated at (U_-, U_+, s) and written in terms of deviations $(\Delta U_-, \Delta U_+, \Delta s)$. Here we introduced the short notations $D_U(\zeta) = D(U(\zeta))$, $F_U(\zeta) = F(U(\zeta))$, $G_U(\zeta) = G(U(\zeta))$, $F_\pm = F(U_\pm)$, and $G_\pm = G(U_\pm)$, where $U(\zeta)$ is the viscous profile of the $S_{i,j}$ shock at the initial point (U_-, U_+, s).

It is remarkable that expression (4.6) does not depend on the choice of the point $U^P = U(\zeta^P)$ (the position of the plane $\mathcal{P}$) in the definition of the function $\mathcal{H}^{add}(U_-, U_+, s)$, just as in the case of Melnikov integrals for systems of planar differential equations [2].

As a model of a physical system, equation (2.1) typically depends on one or more problem parameters. Under variations of these parameters, the functions $G(U)$, $F(U)$, and $D(U)$ undergo perturbations $\Delta G(U)$, $\Delta F(U)$, and $\Delta D(U)$. If these perturbations are small, the manifold $S_{i,j}$ undergoes a small perturbation. The first order approximation of the perturbed manifold can be determined as follows (see the proof in the Appendix).

Theorem 2. *Let $(U_-, U_+, s) \in \mathcal{S}_{i,j}$ and consider perturbations $\Delta G(U)$, $\Delta F(U)$, $\Delta D(U)$ of the system functions. Then the first order approximation of the perturbed manifold $\mathcal{S}_{i,j}$ near the point (U_-, U_+, s) is given by the equations*

$$\Delta \mathcal{H} = -\Delta F_+ + \Delta F_- + s(\Delta G_+ - \Delta G_-), \tag{4.7}$$

$$\begin{aligned} \Delta \mathcal{H}^{add} = &- \int_{-\infty}^{+\infty} \hat{W}^T D_U^{-1} \Delta D_U \, D_U^{-1} \big(F_U - F_- - s(G_U - G_-) \big) d\zeta \\ &+ \int_{-\infty}^{+\infty} \hat{W}^T D_U^{-1} \big(\Delta F_U - \Delta F_- - s(\Delta G_U - \Delta G_-) \big) d\zeta, \end{aligned} \tag{4.8}$$

where $\Delta \mathcal{H}$ and $\Delta \mathcal{H}^{add}$ are given by expressions (4.5) and (4.6).

Theorems 1 and 2 determine all nearby $S_{i,j}$ shock waves, even when problem parameters are changed, using the information on a particular shock and its viscous profile. This method is useful for constructing solutions of conservation laws possessing $S_{i,j}$ shocks, continuation procedures, and parametric analysis.

The characteristic shock waves $S_{i,j}^-$, $S_{i,j}^+$, and $S_{i,j}^\pm$ are studied in the same way. In addition to equations (3.1) and (3.2), (4.3), one should impose conditions ensuring that the shock speed is equal to the corresponding characteristic speed at one or both sides of the shock. This adds one or two equations in Theorems 1 and 2 found by linearizing the equations $s = \lambda(U_-)$ and $s = \lambda(U_+)$.

5 Dual-family shocks in Riemann solutions

The basic initial-value problem for a system of conservation laws (2.2) is the Riemann problem, given by piecewise constant initial data with a single jump at $x = 0$: $U(x, 0) = U_L$ for $x < 0$ and $U(x, 0) = U_R$ for $x > 0$. The solution is found in scale-invariant form $U(x, t) = \hat{U}(\xi)$, $\xi = x/t$, consisting of continuously changing waves (rarefaction waves), jump discontinuities (shock waves), and separating constant states. Classically, there are n families of rarefaction waves, one for each characteristic speed, which we denote by R_i, $i = 1, \ldots, n$. We require all the shock waves to have a viscous profile, i.e., there can be $S_{i,j}$, $S^-_{i,j}$, $S^+_{i,j}$, and $S^\pm_{i,j}$ shocks.

The structure of a Riemann solution is given by a sequence of waves w_k

$$w_1,\ w_2,\ \ldots,\ w_m, \tag{5.1}$$

appearing with increasing value of ξ. Here each wave $w_k \in \{R_i, S_{i,j}, S^-_{i,j}, S^+_{i,j}, S^\pm_{i,j}\}$ is a rarefaction or shock. The wave w_k has left and right states $U_{(k)-}$ and $U_{(k)+}$ and speeds $\xi_{(k)-} < \xi_{(k)+}$ for a rarefaction wave and $s_{(k)} = \xi_{(k)-} = \xi_{(k)+}$ for a shock wave. The left state of the first wave w_1 and the right state of the last wave w_m are the initial conditions of Riemann problem: $U_{(1)-} = U_L$ and $U_{(m)+} = U_R$. The natural requirements in sequence (5.1) are

$$U_{(k)+} = U_{(k+1)-}, \quad \xi_{(k)+} \leq \xi_{(k+1)-}. \tag{5.2}$$

Relations (5.2) imply that the right state of the wave w_k coincides with the left state of the wave w_{k+1}, and the right speed of w_k is lower or equal to the left speed of w_{k+1}. If $\xi_{(k)+} < \xi_{(k+1)-}$ then there is a separating constant state between w_k and w_{k+1}. In this case we will use the notation w_k — w_{k+1}. If $\xi_{(k)+} = \xi_{(k+1)-}$ then the waves do not possess a separating constant state. This situation will be denoted by $w_k \,|\, w_{k+1}$.

A classical Riemann solution consists of n classical wave groups separated by constant states; each classical wave group consists of adjoining rarefactions R_i and classical shocks $S_{i,i}$ (simply S_i) of the same family. The classical structure R_1 — R_2 — S_3 of a Riemann solution in a system of three conservation laws is shown in Figure 5.1(a) using characteristic lines in the space-time plane (shock waves are presented by bold lines and rarefaction waves are given by thin line fans).

Conditions (5.2) imply that each pair of subsequent waves w_k, w_{k+1} in a general Riemann solution has one of the following types

$$\begin{aligned} \{R_j \text{ or } S^*_{i,j}\} &\,—\, \{R_{i'} \text{ or } S^*_{i',j'}\}, \quad j < i', \\ R_i &\;\;|\;\; \{S^-_{i,j} \text{ or } S^\pm_{i,j}\}, \\ \{S^+_{i,j} \text{ or } S^\pm_{i,j}\} &\;\;|\;\; R_j, \end{aligned} \tag{5.3}$$

where $S^*_{i,j}$ stands for $S_{i,j}$, $S^-_{i,j}$, $S^+_{i,j}$, or $S^\pm_{i,j}$. The most important structures of a Riemann solution are the generic ones: they do not change under perturbations of initial conditions U_L, U_R, flux function, and viscosity matrix. Generic structures are "full" in the sense that no wave can be added to a sequence without violating conditions (5.3). For example, the sequence R_1— S_3 is not generic since it can be extended to R_1— S_2— S_3. Only the shocks with $i \geq j$ may appear in generic structures. Generically, overcompressive shocks ($i < j$) bifurcate to a set of waves under perturbations with arbitrarily small amplitudes. Let us assume that all the states of a Riemann solution belong to the region of strict hyperbolicity, where all the characteristic speeds are distinct; viscous profiles of shock waves are not restricted to this region, i.e., they may cross elliptic regions in state space. Then, we can describe generic structures of a Riemann solution (for the case of two conservation laws the following theorem was proved in [12]; for the complete proof see [9]).

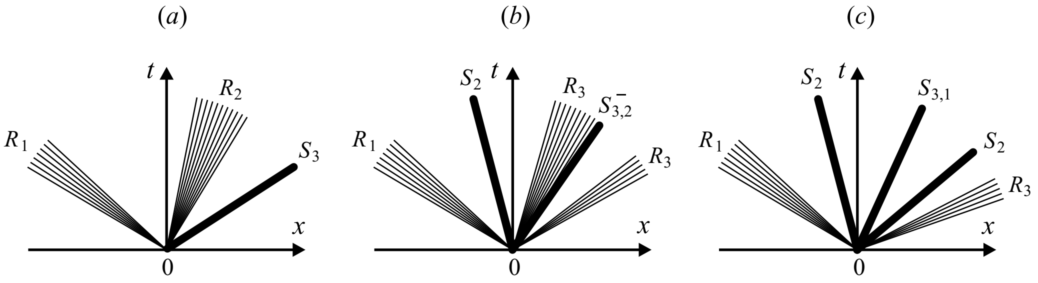

Figure 5.1: Riemann solutions: (a) classical, (b) with $S^-_{3,2}$ shock, (c) with $S_{3,1}$ shock.

Theorem 3. *Let (5.1) be the generic structure of a Riemann solution. Then* $w_1 \in \{R_1,\, S_1,\, S_1^+\}$*,* $w_m \in \{R_n,\, S_n,\, S_n^-\}$*, and each pair* $w_k,\, w_{k+1}$ *has one of the types*

$$\begin{aligned} \{R_j \text{ or } S_{i,j}\} \;—\; \{R_{i'} \text{ or } S_{i',j'}\}, &\quad i' = j+1,\ i \geq j,\ i' \geq j', \\ R_i \;\mid\; \{S^-_{i,j} \text{ or } S^{\pm}_{i,j}\}, &\quad i \geq j, \\ \{S^+_{i,j} \text{ or } S^{\pm}_{i,j}\} \;\mid\; R_j, &\quad i \geq j. \end{aligned} \tag{5.4}$$

As an example, we list two generic nonclassical Riemann solution structures:

$$R_1 — S_2 — R_3 \,|\, S^-_{3,2} — R_3, \tag{5.5}$$

$$R_1 — S_2 — S_{3,1} — S_2 — R_3. \tag{5.6}$$

A distinctive feature of structures (5.5) and (5.6) is that the classical waves R_3 and S_2 appears twice. Riemann solutions with these structures are shown in Figure 5.1(b,c).

We see that Riemann solutions with dual-family shock waves violate the customary classical structure of sequences of n classical wave groups separated by constant states with increasing family number from left to right. The shocks with $i > j+1$ introduce a "jump back" capability in this sequence allowing classical waves of $(j+1), \ldots, (i-1)$-th characteristic families to appear repeatedly. Moreover, from the theoretical point of view, there is no general bound on the number of separated classical waves or of nonclassical shock waves in a Riemann solution for systems of $n > 2$ conservation laws. The existence of several separated waves corresponding to the same characteristic family is a property of Riemann solutions that was observed only in [8].

We remark that the generic structures of Riemann solutions described above do not include transitional rarefaction waves, see [4]. These waves are related to characteristic speeds given by multiple eigenvalues. Therefore, transitional rarefactions do not appear in strictly hyperbolic systems.

6 Dual-family shocks in multi-phase flows in porous media

Let us consider one-dimensional horizontal flow of $n+1$ immiscible fluid phases in a porous medium. The fluids can be, for instance, a mixture of gas or CO_2, water, light oil, and heavy (viscous) oil. We assume that the whole pore space is occupied by the fluids; compressibility, thermal and gravitational effects are neglected. The equations expressing conservation of mass of the i-th phase based on Darcy's law of force is (see e.g. [1])

$$\frac{\partial}{\partial t}(\phi s_i) + \frac{\partial}{\partial x}(v f_i) = \frac{\partial}{\partial x}\left(K l_i \sum_{j\neq i} f_j \frac{\partial p_{ij}}{\partial x}\right), \quad i = 1, \dots, n+1, \tag{6.1}$$

where the constants ϕ and K denote the porosity and absolute permeability of the porous medium, and v is the total seepage velocity of the fluid. For phase i, s_i is the saturation, f_i is the fractional flow function, and l_i is the relative mobility, which can be chosen as the quadratic Corey model (see [1]):

$$f_i = \frac{l_i}{l}, \quad l_i = \frac{s_i^2}{\mu_i}, \quad l = l_1 + \cdots + l_{n+1}, \tag{6.2}$$

where μ_i is the viscosity of phase i; for simplicity, all irreducible phase saturations were set to zero. The capillarity pressures $p_{ij} = p_i - p_j$ between the phases i and j are measured experimentally as functions of saturations; here p_i and p_j are the pressures in phases i and j.

Since the fluids occupy the whole available space, the saturations satisfy

$$s_1 + \cdots + s_n + s_{n+1} = 1. \tag{6.3}$$

Similarly, $f_1 + \cdots + f_n + f_{n+1} = 1$. As a consequence, any n saturations describe the state of the fluid. Hence, any of the $n+1$ equations in system (6.1) is redundant, and the latter can be reduced to an n equation system in n saturations.

As the total seepage velocity v is given by boundary conditions, we assume that it is a positive constant and we set $t = (\phi L/v)\tilde{t}$ and $x = L\tilde{x}$, where L is the characteristic length of the system. Dividing both sides by v/L, this change of variables removes v and ϕ from the left-hand side of system (6.1). For simplicity of notation, we drop the tildes below. Finally, choosing any n saturations as state variables (e.g. $U_i = s_i$, $i = 1, \dots, n$), we arrive at the dimensionless system

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(D(U)\frac{\partial U}{\partial x}\right), \quad U = (U_1, \dots, U_n)^T. \tag{6.4}$$

The components of the vector $F(U) = (F_1, \dots, F_n)^T$ represent the fractional flow functions

$$F_i(U) = \frac{l_i}{l}, \quad l_i(U_i) = U_i^2/\mu_i, \quad l(U) = l_1 + \cdots + l_n + l_{n+1}, \tag{6.5}$$

where

$$U_{n+1} = 1 - U_1 - \cdots - U_n \tag{6.6}$$

is the saturation of remaining $(n+1)$-th phase, $\varepsilon = K/L^2$, and $D(U)$ is the dimensionless viscosity matrix with components $d_{ij}(U) = (L/v) l_i \sum_{k\neq i} f_k \frac{\partial p_{ik}}{\partial x_j}$. The

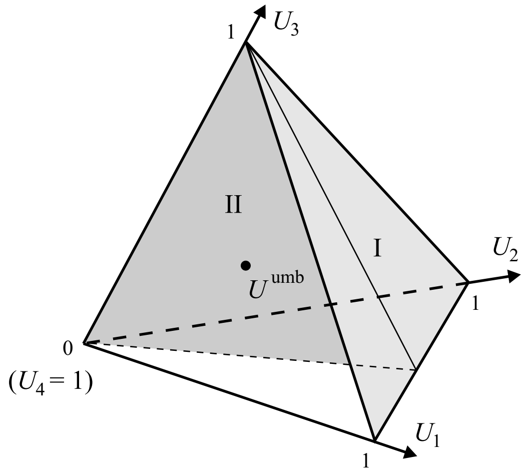

Figure 6.1: State space for four-phase flows. The system restricted to the shaded planes is reduced to a lower dimensional one.

dimensionless parameter ε is small for large characteristic lengths L, so that many aspects of the asymptotic behaviour of solutions may be described by the system of first order conservation laws obtained by taking $\varepsilon = 0$. For simplicity, we will use dimensionless viscosities $\tilde{\mu}_i = \mu_i/(\mu_1 + \cdots + \mu_n + \mu_{n+1})$ in (6.5), so that, dropping the tildes,

$$\mu_1 + \cdots + \mu_n + \mu_{n+1} = 1. \tag{6.7}$$

The physical domain for the state vector U is given by the inequalities

$$U_1 + \cdots + U_n \leq 1, \quad U_i \geq 0, \; i = 1, \ldots, n. \tag{6.8}$$

This domain represents a simplex in state space with $n+1$ vertices; each vertex corresponds to single phase fluid with $U_i = 1$ ($i = 1, \ldots, n+1$); see Figure 6.1.

From (6.5), the $n \times n$ Jacobian matrix $\partial F/\partial U$ can be expressed in the form

$$\begin{aligned} \frac{\partial F}{\partial U} &= \frac{1}{l} \operatorname{diag}\left(\frac{dl_1}{dU_1}, \ldots, \frac{dl_n}{dU_n}\right) \\ &\quad - \frac{1}{l^2} (l_1, \ldots, l_n)^T \left(\frac{dl_1}{dU_1} - \frac{dl_{n+1}}{dU_{n+1}}, \ldots, \frac{dl_n}{dU_n} - \frac{dl_{n+1}}{dU_{n+1}}\right). \end{aligned} \tag{6.9}$$

One can check that all the eigenvalues of matrix (6.9) are strictly positive inside the physical domain (6.8), i.e., flows with positive speed v have only positive characteristic speeds.

There is a so-called umbilic point U^{umb}, at which

$$\frac{dl_1}{dU_1} = \cdots = \frac{dl_n}{dU_n} = \frac{dl_{n+1}}{dU_{n+1}}. \tag{6.10}$$

This is the resonance state, where $\partial F/\partial U$ is a multiple of the identity matrix and, thus, all the characteristic speeds (eigenvalues) λ merge to the same value $\frac{1}{l}\frac{dl_i}{dU_i}$. There are $n+1$ umbilic points at the boundary of the physical domain, which are the vertices of the simplex (6.8). At each vertex, all the characteristic speeds merge to the value zero.

One can check that for the flux functions (6.5) there is a unique umbilic point $U^{\text{umb}} = (\mu_1, \ldots, \mu_n)^T$, $U_{n+1} = \mu_{n+1}$ in the interior of physical domain (6.8), at which all the characteristic speeds equal 2. The umbilic point U^{umb} exists and is unique for the more general case when the relative mobilities $l_i(U_i)$ are arbitrary functions of the corresponding phase saturations U_i with positive first and second derivatives such that $l_1(0) = \ldots = l_{n+1}(0) = 0$ and $dl_1/dU_1(0) = \ldots = dl_{n+1}/dU_{n+1}(0) = 0$. Indeed, the vector $(l_1, \ldots, l_n)$ has positive components inside the physical domain. Hence, the matrix (6.9) is a multiple of the identity matrix if and only if the vector $\left(\frac{dl_1}{dU_1} - \frac{dl_{n+1}}{dU_{n+1}}, \ldots, \frac{dl_n}{dU_n} - \frac{dl_{n+1}}{dU_{n+1}}\right) = 0$, which yields equations (6.10). Since dl_i/dU_i are strictly increasing functions of U_i vanishing at $U_i = 0$, the equations $dl_1/dU_1 = \cdots = dl_n/dU_n = \alpha$ have a unique solution for $\alpha > 0$, and $U_1(\alpha), \ldots, U_n(\alpha)$ are increasing functions of α. The function dl_{n+1}/dU_{n+1}, where $U_{n+1}(\alpha) = 1 - U_1(\alpha) - \cdots - U_n(\alpha)$, is a decreasing function of α. Additionally, we have $dl_{n+1}/dU_{n+1} > 0$ at $\alpha = 0$ ($U_{n+1} = 1$), and $dl_{n+1}/dU_{n+1} = 0$ at $\alpha = \alpha^* > 0$, where the value of α^* is given by the equation $U_{n+1}(\alpha) = 0$. Hence, there exists a unique α^{umb} solving the equation $dl_{n+1}/dU_{n+1} = \alpha$ such that $0 < \alpha^{\text{umb}} < \alpha^*$. This yields a unique umbilic point $U^{\text{umb}} = (U_1(\alpha^{\text{umb}}), \ldots, U_n(\alpha^{\text{umb}}))^T$, which lies inside the physical region.

In order to study dual-family shock waves, we artificially take the identity viscosity matrix $D(U) \equiv I$. Our main motivation here is to show analytically the existence of all types of $S_{i,j}$ shock waves with $-n < i - j < n$ in the system. This existence provides the evidence that any $S_{i,j}$ shock may appear in the system with a realistic viscosity matrix $D(U)$.

6.1 Reduced dimension systems

By setting one of the $U_i = 0$ in equation (6.4), we obtain a reduced system describing n instead of $n + 1$ phase flows. This system "lives" on the face of the simplex; see Figure 6.1, where the lightly shaded face I corresponds to $U_4 = 0$. Of course, all the results of this section hold for the reduced system. This reduction can be done iteratively until we reach a scalar partial differential equation describing two-phase flow. This system is restricted to one of the edges of the simplex. Notice that this reduction is valid for any physical viscosity matrix $D(U)$.

There are other subsystems of (6.4) with dimension $n - 1$. Let us consider the simplex of codimension 1:

$$U = \left(\frac{\mu_1}{\mu_1 + \mu_2}\,\rho,\ \frac{\mu_2}{\mu_1 + \mu_2}\,\rho,\ U_3, \ldots, U_n\right)^T;\quad \rho + U_3 + \cdots + U_n \leq 0;\ \rho, U_3, \ldots, U_n \geq 0. \tag{6.11}$$

For $n = 3$ this is the shaded plane II containing U^{umb} shown in Figure 6.1. One can check that the first two equations of system (6.4), (6.5) restricted to plane (6.11) are equivalent (in this case we consider $D(U) \equiv I$), and the system possesses two equal characteristic speeds

$$\lambda = \frac{2\rho}{(\mu_1+\mu_2)l}. \tag{6.12}$$

As a result, system (6.4) is reduced to the $n-1$ dimensional system with state vector $(\rho, U_3, \ldots, U_n)^T$ and viscosities $\mu_1+\mu_2$, $\mu_3, \ldots, \mu_n$. This system describes the flow of phases $U_3, \ldots, U_n$ and the mixture of phases U_1, U_2 in the proportion $U_1/U_2 = \mu_1/\mu_2$ that acts as a single phase. Analogous reductions can be done taking any two saturations U_i and U_j instead of U_1 and U_2.

Repeating such a reduction several times, we can obtain systems of lower dimensions. Each of these systems describes a multi-phase flow. Thus, all the results of this section hold for any of the reduced systems. Notice that the reduced systems "live" on lower dimensional simplices in the interior of physical state space (6.8).

Using reduction (6.11) $n-1$ times, each time with two state variables, we arrive at the system restricted to the line

$$U_\rho = \left(\frac{\mu_1}{\mu}, \ldots, \frac{\mu_n}{\mu}\right)^T \rho, \quad \mu = \mu_1 + \cdots + \mu_n. \tag{6.13}$$

Up to multiplication by the constant μ_i/μ, all n equations in system (6.4) are equivalent to the scalar partial differential equation on the line (6.13):

$$\frac{\partial \rho}{\partial t} + \frac{\partial F(\rho)}{\partial x} = \varepsilon \frac{\partial^2 \rho}{\partial x^2}, \tag{6.14}$$

where

$$F(\rho) = \frac{\rho^2}{\mu l}, \quad l(\rho) = \frac{\rho^2 - 2\mu\rho + \mu}{\mu(1-\mu)}. \tag{6.15}$$

One can check that equation (6.14) coincides with the viscous profile equation taken for a scalar conservation law describing two-phase flow, where $U = U_1 = \rho$, $U_2 = 1-\rho$, $\mu_1 = \mu$, and $\mu_2 = 1-\mu$.

According to (6.8), the physical interval for ρ is $0 \le \rho \le 1$. The line (6.13) contains the point U^{umb} at $\rho = \mu$, as well as the vertex $U = 0$ at $\rho = 0$. At points with $0 < \rho < \mu$ or $\mu < \rho \le 1$, there are $n-1$ equal characteristic speeds

$$\lambda(\rho) = \frac{2\rho}{\mu l}. \tag{6.16}$$

There is an $(n-1)$-dimensional eigenspace of the matrix $\partial F/\partial U$ corresponding to the multiple eigenvalue $\lambda(\rho)$, which consists of the vectors $r = (r_1, \ldots, r_n)^T$ such that $r_1 + \ldots + r_n = 0$. The remaining characteristic speed is equal to

$$\tilde{\lambda}(\rho) = \frac{2\rho(1-\rho)}{\mu(1-\mu)l^2}, \tag{6.17}$$

with corresponding eigenvector $\tilde{r} = (\mu_1, \ldots, \mu_n)^T$, which is parallel to the line (6.13).

6.2 $S_{n,1}$ dual-family shock waves

In this subsection, we seek for solutions of the viscous profile equation (2.3) that have the form (6.13). Here $\rho(\zeta)$ is a function of ζ such that $\rho_\pm = \rho(\pm\infty)$ and $U_\pm = U_{\rho_\pm}$. Along the line (6.13), all n equations in system (2.3) are equivalent to the scalar ordinary differential equation

$$\dot{\rho} = \frac{\rho^2}{\mu l} - \frac{\rho_-^2}{\mu l_-} - s(\rho - \rho_-), \quad l_- = l(\rho_-), \tag{6.18}$$

which is the viscous profile equation for two-phase flow described by (6.14). Since the right-hand side of (6.18) vanishes at $\rho = \rho_+$, we find the shock speed

$$s = \frac{\rho_- + \rho_+ - 2\rho_-\rho_+}{\mu(1-\mu)l_- l_+}, \quad l_\pm = l(\rho_\pm). \tag{6.19}$$

Here expression (6.15) for $l_\pm$ was used. One can show that the ordinary differential equation (6.18) has a solution $\rho(\zeta)$ such that $\rho(\pm\infty) = \rho_\pm$ provided that both inequalities

$$\begin{aligned} (\rho_+ - \rho_-)(\rho_-^2 + 2\rho_-\rho_+ - 2\rho_-^2\rho_+ - \mu) &> 0, \\ (\rho_+ - \rho_-)(\rho_+^2 + 2\rho_-\rho_+ - 2\rho_-\rho_+^2 - \mu) &> 0 \end{aligned} \tag{6.20}$$

hold. Conditions (6.20) are equivalent to the inequalities

$$\tilde{\lambda}_+ < s < \tilde{\lambda}_-, \quad \tilde{\lambda}_\pm = \tilde{\lambda}(\rho_\pm), \tag{6.21}$$

where $\tilde{\lambda}$ is the distinct characteristic speed (6.17).

Under the conditions (2.7) and (6.21), we see that the shock wave having the viscous profile U_ρ is of type $S_{n,1}$ exactly if

$$\lambda_- < s < \lambda_+, \quad \lambda_\pm = \lambda(\rho_\pm), \tag{6.22}$$

where $\lambda(\rho)$ is the characteristic speed (6.16) with multiplicity $n-1$. With the use of (6.15) and (6.19), conditions (6.22) reduce to

$$\begin{aligned} \mu\rho_+ - \mu\rho_- + 2\rho_-\rho_+(\mu - \rho_+) &> 0, \\ \mu\rho_+ - \mu\rho_- - 2\rho_-\rho_+(\mu - \rho_-) &> 0. \end{aligned} \tag{6.23}$$

Figure 6.2 shows the region corresponding to the values (ρ_-, ρ_+) satisfying inequalities (6.20) and (6.23) for $0 \le \mu \le 1$. The constants ρ_- and ρ_+ taken in this region define left and right states and speeds of $S_{n,1}$ shocks by formulae (6.13) and (6.19). One can see that $S_{n,1}$ shock waves exist in the system for any μ. If $\mu \ge 1/2$, these shocks can be arbitrarily small, since the boundary of the region contains the point $\rho_- = \rho_+ = \mu$ corresponding to the umbilic point. If $\mu < 1/2$, the shock amplitude $\|U_+ - U_-\|$ is bounded away from zero. In all the cases we have $\rho_- < \mu < \rho_+$, which means that the viscous profile passes through the umbilic point.

Since the state variables $U_1, \ldots, U_n$ can be chosen as any n out of $n+1$ saturations $s_1, \ldots, s_{n+1}$, there are n other lines in state space similar to (6.13) containing the left and right states of $S_{n,1}$ shocks.

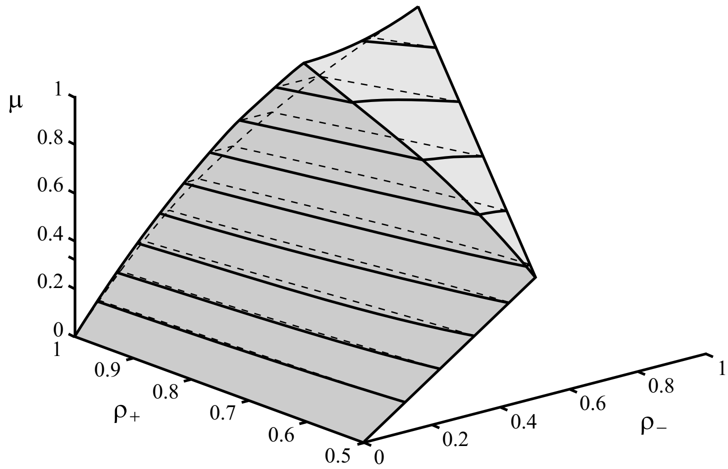

Figure 6.2: Values of (ρ_-, ρ_+) corresponding to $S_{n,1}$ shocks for different μ.

Characteristic shocks of types $S^-_{n,1}$, $S^+_{n,1}$, and $S^\pm_{n,1}$ can be found, respectively, when the first, second, or both inequalities (6.20) become equalities. These equalities correspond to $s = \tilde{\lambda}_-$, $s = \tilde{\lambda}_+$, and $s = \tilde{\lambda}_- = \tilde{\lambda}_+$, respectively, where $\tilde{\lambda}_\pm$ are simple characteristic speeds at different sides of the shock. It turns out that there are no $S^+_{n,1}$ and $S^\pm_{n,1}$ shocks with viscous profile (6.13). Parameters ρ_- and ρ_+ corresponding to $S^-_{n,1}$ shocks belong to the front face of the boundary shown darker in Figure 6.2.

According to the results of Section 5, a total of $n-1$ classical waves may be present at both sides of $S_{n,1}$ shock in a Riemann solution. Thus, there are Riemann solutions with $S_{n,1}$ shocks for any initial conditions U_L and U_R taken in certain neighborhoods of the left and right shock states U_- and U_+. An example of a structure of Riemann solutions with $S_{3,1}$ shocks was given in (5.6).

6.3 General $S_{i,j}$ dual-family shocks

By reducing system dimension, we can find other types of $S_{i,j}$ shocks. First, let us consider the case when

$$U_1 = \cdots = U_k = 0, \tag{6.24}$$

i.e., only the phases $U_{k+1}, \ldots, U_n, U_{n+1}$ are present. Thus, (6.4) reduces to the system for $(n-k+1)$-phase flow. Assume that we found an $S_{i,j}$ shock with speed s in the reduced system. In the full $(n+1)$-phase system, this corresponds to a shock for which the first k components of the viscous profile are zero. Due to (6.24), there are k zero characteristic speeds at both sides of the shock. Thus, such a shock will be an $S_{i+k,j+k}$ shock for the full system. The shock speed s is always positive, which follows from the positivity of characteristic speeds.

In the previous subsection, we found $S_{n,1}$ shocks in a general $(n+1)$-phase system. Hence, by using the described recursive relation, we find an $S_{n,k+1}$ dual-family shock for any $0 \le k < n$ with the viscous profile

$$U(\zeta) = \left(0, \ldots, 0, \frac{\mu_{k+1}}{\mu}, \ldots, \frac{\mu_n}{\mu}\right)^T \rho(\zeta), \quad \mu = \mu_{k+1} + \cdots + \mu_n. \tag{6.25}$$

Here $(\mu_{k+1},\dots,\mu_n)^T\rho(\zeta)/\mu$ is the viscous profile of the $S_{n-k,1}$ shock found for the reduced $(n-k+1)$-phase system. These shocks lie on the boundary of the physical domain (6.8), yet they determine certain points (U_-,U_+,s) on the shock surfaces $\mathcal{S}_{n,k+1}$. By using Theorem 1 in Section 4, we can find all nearby shock waves of the same type. Some of them belong to the interior of the physical domain (6.8) and, thus, they represent $S_{n,k+1}$ shocks intrinsic to $(n+1)$-phase flow.

The second type of dimension reduction corresponds to the plane (6.11). As shown above, in this plane the system reduces to an $n-1$ dimensional system describing n phase flows. There are two equal characteristic speeds (6.12) at each point of the plane. Again, let us consider an $S_{i,j}$ shock with speed s in the reduced system. Then, if $\lambda_- < s < \lambda_+$ for double characteristic speeds $\lambda_\pm$, we get an $S_{i+1,j}$ shock in the full $(n+1)$-phase system. If $\lambda_+ < s < \lambda_-$, this shock becomes an $S_{i,j+1}$ shock in the full system. Finally, if $s < \lambda_\pm$ or $s > \lambda_\pm$, we obtain $S_{i,j}$ and $S_{i+1,j+1}$ shocks, respectively. This reduction can be used repeatedly. For example, $S_{n,1}$ shocks previously described are obtained by using $n-1$ repeated reductions.

The described approach allows finding dual-family shocks $S_{i,j}$ for any $0 < i-j < n$. Note that by taking opposite signs in inequalities (6.21), (6.22) (equivalently, in (6.20), (6.23)), we obtain overcompressive shocks $S_{1,n}$. With these shocks, by using the dimension reduction approach, we can locate overcompressive $S_{i,j}$ shocks for any $-n < i-j < 0$. The same method can be used for finding particular viscous profiles for these shocks (recall that overcompressive shocks possess an infinite number of viscous profiles).

6.4 Sensitivity analysis and continuation method

Let us apply Theorems 1 and 2 for an $S_{n,1}$ shock with the viscous profile (6.13). For this purpose, we fix some values of ρ_- and ρ_+, which determine uniquely U_-, U_+, s, and the viscous profile $U(\zeta) = \left(\frac{\mu_1}{\mu},\dots,\frac{\mu_n}{\mu}\right)^T\rho(\zeta)$. With the use of expression (6.9), the matrix $B(U(\zeta),s)$ defined in (2.5) takes the form

$$B(U(\zeta),s) = \left(\frac{2\rho(\zeta)}{\mu l(\rho(\zeta))} - s\right) I + \frac{2(\mu-\rho(\zeta))}{\mu(1-\mu)}\left(\frac{\rho(\zeta)}{\mu l(\rho(\zeta))}\right)^2 (\mu_1,\dots,\mu_n)^T(1,\dots,1). \tag{6.26}$$

Then, the general solution $W(\zeta)$ of the adjoint linear system (4.2) is found as

$$W(\zeta) = \left(\frac{w_1}{\mu_1},\dots,\frac{w_n}{\mu_n}\right)^T \eta(\zeta), \tag{6.27}$$

where $w_1,\dots,w_n$ are arbitrary constants satisfying the condition $w_1+\cdots+w_n = 0$, and the scalar function $\eta(\zeta)$ is determined by the equation

$$\dot\eta = \left(s - \frac{2\rho(\zeta)}{\mu l(\rho(\zeta))}\right)\eta, \quad \int_{-\infty}^{+\infty}\eta(\zeta)d\zeta = 1. \tag{6.28}$$

The latter equality in (6.28) is the normalization condition. The solution $\eta(\zeta)$ of (6.28) exist due to relations (6.16) and (6.22). Taking $n-1$ linearly independent solutions as columns of the $n \times (n-1)$ matrix $\hat{W}(\zeta)$, we obtain

$$\hat{W}(\zeta) = \hat{W}_0\,\eta(\zeta), \quad \hat{W}_0 = \begin{pmatrix} 1/\mu_1 & 1/\mu_1 & \cdots & 1/\mu_1 \\ -1/\mu_2 & 0 & \cdots & 0 \\ 0 & -1/\mu_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1/\mu_n \end{pmatrix}. \tag{6.29}$$

Using matrix (6.29) in Theorem 1, we find the differentials

$$\Delta\mathcal{H} = \left(\frac{\partial F}{\partial U} - sI\right)_{U=U_+} \Delta U_+ - \left(\frac{\partial F}{\partial U} - sI\right)_{U=U_-} \Delta U_- - (U_+ - U_-)\Delta s, \tag{6.30}$$

$$\Delta\mathcal{H}^{add} = \left(\frac{2\rho_-}{\mu l_-} - s\right)\hat{W}_0^T \Delta U_-. \tag{6.31}$$

Condition $\Delta\mathcal{H}^{add} = 0$ with expression (6.31) yields $\Delta U_- = \left(\frac{\mu_1}{\mu}, \ldots, \frac{\mu_n}{\mu}\right)^T \Delta\rho_-$. Then using (6.9) and (6.13) in (6.30), one can check that $\Delta U_+ = \left(\frac{\mu_1}{\mu}, \ldots, \frac{\mu_n}{\mu}\right)^T \Delta\rho_+$ and the expression for Δs coincides with the linearization of (6.19). This shows that the approximation of $S_{n,1}$ shock states and speeds obtained by Theorem 1 agrees with the analytical results.

Now let us assume that the viscosity matrix $D(U) \equiv I$ suffers a small variation $\Delta D(U)$, while the functions $G(U) = U$ and $F(U)$ remain unchanged. Then the change of states and speeds of $S_{n,1}$ shocks can be found using Theorem 2 as

$$\Delta\mathcal{H} = 0, \tag{6.32}$$

$$\Delta\mathcal{H}^{add} = -\int_{-\infty}^{+\infty} \hat{W}_0^T \eta\,\Delta D_U \left(\frac{\mu_1}{\mu}, \ldots, \frac{\mu_n}{\mu}\right)^T \left(\frac{\rho^2}{\mu l} - \frac{\rho_-^2}{\mu l_-} - s(\rho - \rho_-)\right), \tag{6.33}$$

where $\Delta D_U = \Delta D(U(\zeta))$. One can see that if $\Delta D(U) = \gamma(U)I$, where $\gamma(U)$ is a scalar function, the right-hand side of (6.33) vanishes. Indeed, one can check analytically that the states and speeds of $S_{n,1}$ shocks remain unchanged for the viscosity matrix $D(U) = \gamma(U)I$ with an arbitrary positive function $\gamma(U)$ (the only change is in equation (6.18) whose left-hand side becomes $\gamma(U_\rho)\dot{\rho}$).

In general, the perturbation $\Delta D(U)$ of the viscosity matrix changes the parameters of $S_{n,1}$ shocks. Since the change of left and right states and speeds can be found using Theorem 2, one can use the continuation method to find $S_{n,1}$ shocks in a system with a particular nontrivial $D(U)$. For this purpose, the matrix $D(U)$ is changed from I to the required value in a sequence of small steps; at each step the viscous profile and the matrix $\hat{W}(\zeta)$ have to be recomputed and used in Theorem 2 for finding approximation for the next step. A similar procedure can be applied to all other types of $S_{i,j}$ shocks. We can expect to find $S_{i,j}$ shocks in the system, at least for viscosity matrices $D(U)$ close to $\gamma(U)I$ with some function $\gamma(U)$ and for small perturbations of flux functions.

We note that the umbilic points give rise to elliptic regions in state space under a general perturbation the flux function $F(U)$. This happens, for instance, in Stone's model for three-phase flow through porous media [17]. Generically, the resonant points in a perturbed system, where all n characteristic merge, will have one-dimensional eigenspace (one eigenvector) unlike the umbilic point U^{umb}, which has n eigenvectors. At the same time, all $S_{i,j}$ shocks persist in the system. Hence $S_{i,j}$ shocks are not necessarily related to the existence of umbilic points.

6.5 Physical applications

In many physical applications of system (6.1), the goal is to study the possibility of recovering one phase of the fluid (e.g. oil) on the right side by injecting different fluid phases (e.g. water and steam) on the left. The variety of $S_{i,j}$ shocks that can appear in the system has to be taken into account in analysis of physical behavior, e.g., when predicting oil recovery in petroleum engineering practice.

By adopting the method of [7], one should be able to show stability of $S_{i,j}$ dual-family shocks with $i > j$ as solutions of equation (2.1), at least for shocks of small amplitudes. We expect that this is true also in case of perturbed flux functions and viscosity matrices.

7 Conclusion

We studied a general class of shock waves satisfying the viscous profile admissibility criterion in general systems of n conservation laws. Roughly speaking, shock waves are classified by comparing their speeds with characteristic speeds at opposite sides of the wave. As a result, we are led to consider dual-family shocks $S_{i,j}$ associated with characteristic families i and j at the left and right sides, respectively. We develop a constructive method for analytical and numerical study of such shocks and their perturbations under change of problem parameters. One remarkable feature of $S_{i,j}$ shocks with $i > j$ is that their left and right states and speeds depend on the viscosity matrix. The other is that these shocks dramatically enlarge the possible structures of generic Riemann problem solutions. In Riemann solutions with such shock waves, classical wave groups of the same characteristic family can appear repeatedly from different sides of dual-family shocks, separated by constant states. Thus, classical wave groups in solutions with dual-family shocks does not necessarily follow in increasing family number. As a wide variety of $S_{i,j}$ shocks is found in systems describing multi-phase flows through porous media, there is a promising perspective for analyzing and using these shocks in physical applications.

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8 Appendix

8.1 Proof of Proposition 1

Let us define the linear spaces $\mathcal{V}_-$ and $\mathcal{V}_+$ of solutions $V(\zeta) \in \mathbb{R}^n$ of the system

$$\dot{V} = B(U(\zeta), s)V, \tag{8.1}$$

vanishing as $\zeta \to -\infty$ and $\zeta \to +\infty$, respectively. Similarly, we define the linear spaces $\mathcal{W}_-$ and $\mathcal{W}_+$ of solutions $W(\zeta) \in \mathbb{R}^n$ of the system

$$\dot{W} = -B^T(U(\zeta), s)W, \tag{8.2}$$

vanishing as $\zeta \to -\infty$ and $\zeta \to +\infty$, respectively. Since the matrix $B(U, s)$ has the eigenvalues $\mu_i(U, s)$, $i = 1, \ldots, n$, and the matrix $-B^T(U, s)$ has the eigenvalues $-\mu_i(U, s)$, $i = 1, \ldots, n$ satisfying inequalities (2.6), we find $\dim \mathcal{V}_- = n - i + 1$, $\dim \mathcal{V}_+ = j$, $\dim \mathcal{W}_- = i - 1$, and $\dim \mathcal{W}_+ = n - j$. The set $\mathcal{W}$ is the intersection $\mathcal{W} = \mathcal{W}_- \cap \mathcal{W}_+$.

Since systems (8.1) and (8.2) are adjoint, the linear space $\mathcal{W}_-$ is the orthogonal complement of $\mathcal{V}_-$ at each value of ζ. Indeed, let $V(\zeta) \in \mathcal{V}_-$ and $W(\zeta) \in \mathcal{W}_-$. Integrating the product $W^T(\zeta)B(U(\zeta), s)V(\zeta)$ in the interval $-\infty < \zeta \le \zeta^*$, we obtain

$$\begin{aligned} \int_{-\infty}^{\zeta^*} W^T(\zeta)B(U(\zeta), s)V(\zeta)d\zeta &= \int_{-\infty}^{\zeta^*} W^T(\zeta)\dot{V}(\zeta)d\zeta \\ &= W^T(\zeta)V(\zeta)\Big|_{-\infty}^{\zeta^*} - \int_{-\infty}^{\zeta^*} \dot{W}^T(\zeta)V(\zeta)d\zeta \\ &= W^T(\zeta^*)V(\zeta^*) + \int_{-\infty}^{\zeta^*} W^T(\zeta)B(U(\zeta), s)V(\zeta)d\zeta. \end{aligned} \tag{8.3}$$

Here, the integration by parts, equations (8.1), (8.2), and the conditions $V(-\infty) = W(-\infty) = 0$ were used. From (8.3), we obtain that

$$W^T(\zeta^*)V(\zeta^*) = 0 \tag{8.4}$$

for any real ζ^*.

Since the tangent space to the manifold $\mathcal{M}_u(U_-)$ at $U(\zeta)$ is given by the vectors $V(\zeta) \in \mathcal{V}_-$, the vector $W(\zeta) \in \mathcal{W}$ is orthogonal to $\mathcal{M}_u(U_-)$ at $U(\zeta)$. Analogously, we can prove that $W(\zeta)$ is orthogonal to $\mathcal{M}_s(U_+)$ at $U(\zeta)$.

If the manifolds $\mathcal{M}_u(U_-)$ and $\mathcal{M}_s(U_+)$ intersect quasi-transversally, we have $\dim(\mathcal{V}_- \cap \mathcal{V}_+) = 1$. Hence,

$$\begin{aligned} \dim \mathcal{W} = \dim(\mathcal{W}_- \cap \mathcal{W}_+) &= n - \dim(\mathcal{V}_- \cup \mathcal{V}_+) \\ &= n - \big(\dim \mathcal{V}_- + \dim \mathcal{V}_+ - \dim(\mathcal{V}_- \cap \mathcal{V}_+)\big) = i - j. \end{aligned} \tag{8.5}$$

8.2 Proof of Theorems 1 and 2

Equations (4.5) and (4.7) are obtained by varying the Rankine–Hugoniot conditions (3.1).

Let $U(\zeta)$ be a viscous profile of an $S_{i,j}$ shock with states $U_\pm$ and speed s. Let us take perturbations of the left and right states $U_\pm + \Delta U_\pm$ and of the speed $s + \Delta s$. These perturbations are assumed to satisfy the Rankine-Hugoniot conditions. Then we can write the variational equation for system (2.3) as

$$\Delta\dot{U} = B(U(\zeta), s)\Delta U - D_U^{-1}\left(\frac{\partial F}{\partial U} - s\frac{\partial G}{\partial U}\right)_{U=U_-}\Delta U_- - D_U^{-1}(G_U - G_-)\Delta s. \quad (8.6)$$

The perturbed solution $U(\zeta)+\Delta U_u(\zeta)$ lying in the unstable manifold $\mathcal{M}_u(U_- + \Delta U_-)$ satisfies the condition

$$\Delta U_u(-\infty) = \Delta U_-. \quad (8.7)$$

Pre-multiplying (8.6) by the transpose of the function $W(\zeta) \in \mathcal{W}$ and integrating in the interval $-\infty < \zeta \leq \zeta^*$, we find

$$\begin{aligned} W^T(\zeta^*)\Delta U_u(\zeta^*) &= -\left(\int_{-\infty}^{\zeta^*} W^T D_U^{-1}\left(\frac{\partial F}{\partial U} - s\frac{\partial G}{\partial U}\right)_{U=U_-} d\zeta\right)\Delta U_- \\ &\quad - \left(\int_{-\infty}^{\zeta^*} W^T D_U^{-1}(G_U - G_-)d\zeta\right)\Delta s. \end{aligned} \quad (8.8)$$

Here, we used integration by parts with equations (8.2), (8.7) and the condition $W(-\infty) = 0$. Analogously, for the perturbed solution $U(\zeta) + \Delta U_s(\zeta)$ lying in the stable manifold $\mathcal{M}_s(U_+ + \Delta U_+)$, i.e., $\Delta U_s(+\infty) = \Delta U_+$, we obtain

$$\begin{aligned} W^T(\zeta^*)\Delta U_s(\zeta^*) &= \left(\int_{\zeta^*}^{+\infty} W^T D_U^{-1}\left(\frac{\partial F}{\partial U} - s\frac{\partial G}{\partial U}\right)_{U=U_-} d\zeta\right)\Delta U_- \\ &\quad + \left(\int_{\zeta^*}^{+\infty} W^T D_U^{-1}(G_U - G_-)d\zeta\right)\Delta s. \end{aligned} \quad (8.9)$$

Subtracting (8.8) from (8.9), we get

$$\begin{aligned} W^T(\zeta^*)(U_+^P - U_-^P) &= W^T(\zeta^*)(\Delta U_s(\zeta^*) - \Delta U_u(\zeta^*)) \\ &= \left(\int_{-\infty}^{+\infty} W^T D_U^{-1}\left(\frac{\partial F}{\partial U} - s\frac{\partial G}{\partial U}\right)_{U=U_-} d\zeta\right)\Delta U_- \\ &\quad + \left(\int_{-\infty}^{+\infty} W^T D_U^{-1}(G_U - G_-)d\zeta\right)\Delta s. \end{aligned} \quad (8.10)$$

Using expression (8.10) in (4.3), we obtain formula (4.6).

Finally, formula (4.8) is derived in the same way from equation (8.6), where the term $-D_U^{-1}\Delta D_U\, D_U^{-1}\big(F_U - F_- - s(G_U - G_-)\big) + D_U^{-1}\big(\Delta F_U - \Delta F_- - s(\Delta G_U - \Delta G_-)\big)$ resulting from variations of the system functions is added to the right-hand side.

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