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ON SINGULARITIES OF A BOUNDARY OF THE STABILITY DOMAIN*

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This paper is dedicated to V. I. Arnold on the occasion of his 60th birthday.

Abstract. This paper deals with the study of generic singularities of a boundary of the stability domain in a parameter space for systems governed by autonomous linear differential equations $\dot{y} = Ay$ or $x^{(m)} + a_1 x^{(m-1)} + \cdots + a_m x = 0$. It is assumed that elements of the matrix A and coefficients of the differential equation of mth order smoothly depend on one, two, or three real parameters. A constructive approach allowing the geometry of singularities (orientation in space, magnitudes of angles, etc.) to be determined with the use of tangent cones to the stability domain is suggested. The approach allows the geometry of singularities to be described using only first derivatives of the coefficients a_i of the differential equation or first derivatives of the elements of the matrix A with respect to problem parameters with its eigenvectors and associated vectors calculated at the singular points of the boundary. Two methods of study of singularities are suggested. It is shown that they are constructive and can be applied to investigate more complicated singularities for multiparameter families of matrices or polynomials. Two physical examples are presented and discussed in detail.

Key words. stability boundary, generic singularity, tangent cone, collapse of the Jordan block, versal deformation

AMS subject classifications. 93D99, 34D20

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Introduction. We consider a system of autonomous linear differential equations $\dot{y} = Ay$ assuming that the real matrix operator A of dimension $m \times m$ smoothly depends on n real parameters. Stability of the trivial solution $y \equiv 0$ of the system is considered. It is well known that the trivial solution is asymptotically stable, if all eigenvalues of A have negative real part, and unstable if at least one of the eigenvalues of A has positive real part. According to this definition the parameter space $\mathbf{R}^n$ is divided into the stability and instability domains. Boundary between these domains corresponds to the cases when some of the eigenvalues have zero real part while other eigenvalues have negative real part.

Arnold [3, 4, 5] listed all the generic singularities arising at the stability boundary in two- and three-dimensional space of parameters and gave their description up to a smooth change of problem parameters (diffeomorphism). In this paper we suggest a constructive approach allowing one to determine the geometry of singularities (orientation in space, magnitudes of angles, etc.) using only first derivatives of the matrix A with respect to parameters and left and right eigenvectors and associated vectors of A, corresponding to the Jordan structure of the matrix A at the singular points of the boundary. Our study is essentially based on the perturbation theory of eigenvalues and eigenvectors, developed by Vishik and Lyusternik [16] and Lidskii [11] and applied by Seyranian [12, 13] to the case of multiple parameters, and the theory of

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normal forms of families of matrices by Arnold [2] and Galin [7]. Investigation of the boundary of the stability domain is closely related to construction of tangent cones to the stability domain at the boundary point, introduced by Levantovskii [9], and the problem of finding stable perturbations of nonsymmetric matrices considered by Burke and Overton [6].

We investigate generic singularities of a boundary of the stability domain for linear autonomous differential equation of mth order $x^{(m)} + a_1 x^{(m-1)} + \cdots + a_m x = 0$, assuming that the coefficients a_i smoothly depend on one, two, or three real parameters. Explicit formulae to describe the geometry of singularities in the parameter space are derived.

As examples, two physical problems are considered: stability of equilibrium of a voltaic arc in an electric circuit and stability of Ziegler's double pendulum with two different damping parameters. In the first problem at the singular point "double zero" we find the angle of the corner of the stability boundary and its orientation in parameter plane. In the second problem it is shown that the singularity, arising at the critical load of the system without damping, represents, according to Arnold's terminology [3, 4], the "deadlock of an edge." This leads to the effects of destabilization due to small damping and absence of a limit of the critical load when damping parameters tend to zero. Similar effects could be expected for systems with singularities like "break of an edge."

The main result of the paper is that the Jordan structure of the matrix A and its first derivatives with respect to problem parameters at any point of the stability boundary define a linear approximation of the stability domain in the vicinity of the considered point. Similar results are valid for stability problems governed by a linear differential equation of mth order: to determine the geometry of the stability domain in the vicinity of a singular point of the boundary we need only to know the multiplicity of the root of the characteristic polynomial and the first derivatives of the coefficients of the differential equation with respect to problem parameters at this point.

1. Collapse of Jordan blocks. Let us consider an eigenvalue problem

$$A\,u = \lambda\,u. \tag{1.1}$$

Here A is a real nonsymmetric square $m \times m$ matrix, the elements of which, $a_{ij}(p)$, $i,\ j = 1,\ 2,\ \ldots,\ m$, are smooth functions of a real vector of parameters $p = (p_1,\ p_2,\ \ldots,\ p_n)^T$; λ is an eigenvalue; and u is a corresponding eigenvector of dimension m.

It is assumed that at fixed $p = p_0$, λ_0 is an eigenvalue of $A(p_0)$, and a change of the eigenvalue λ_0 is sought that depends on a change of the vector of parameters p. For this purpose let us consider a perturbation of the vector p_0 in the form $p = p(\varepsilon)$, $p(0) = p_0$, where ε is a small positive number and $p(\varepsilon)$ is a smooth function of ε. Determine a real vector of direction $e = (e_1,\ e_2,\ \ldots,\ e_n) = dp/d\varepsilon \neq 0$, where the derivative is calculated at $\varepsilon = 0$. As a result the matrix A takes the increment

$$A(p(\varepsilon)) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \tag{1.2}$$

where the matrices A_0 and A_1 are given by the relations

$$A_0 = A(p_0), \quad A_1 = \sum_{s=1}^{n} \frac{\partial A}{\partial p_s}\, e_s. \tag{1.3}$$

The derivatives in (1.3) are taken at $p = p_0$.

Due to the perturbation of the vector p_0, the eigenvalue λ_0 and the eigenvector u_0 take increments. According to the perturbation theory of non-self-adjoint operators, developed in [16, 11], these increments can be expressed as series in integer or fractional powers of ε, depending on the Jordan structure corresponding to the eigenvalue λ_0. A multiple eigenvalue λ_0 generally splits into l simple eigenvalues under the perturbation of parameters $p = p(\varepsilon)$. Expansions for eigenvalues and eigenvectors contain terms with fractional powers $\varepsilon^{j/l}$, $j = 0, 1, 2 \ldots$, where l is the length of the Jordan chain [16, 11].

1.1. Simple eigenvalue. Assume that λ_0 is a simple eigenvalue of the matrix A_0 and u_0 is the corresponding eigenvector. In this case expansions of λ and u take the form [16, 11]

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon\,\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \\ u &= u_0 + \varepsilon\,w_1 + \varepsilon^2 w_2 + \cdots. \end{aligned} \tag{1.4}$$

For the following presentation we also need the left eigenvector v_0, corresponding to λ_0,

$$v_0^T A_0 = \lambda_0\, v_0^T. \tag{1.5}$$

The eigenvectors u_0 and v_0 in the case of simple eigenvalue λ_0 are related by the condition $v_0^T u_0 \neq 0$. Substituting (1.2), (1.4) into (1.1) and using (1.5) we find [16, 11]

$$\lambda_1 = \frac{v_0^T A_1 u_0}{v_0^T u_0}. \tag{1.6}$$

This expression with the use of (1.3) can be given in the form

$$\lambda_1 = (r,\, e) + i\,(k,\, e), \tag{1.7}$$

where brackets denote the scalar product in $\mathbf{R}^n$, i.e., $(a,\, b) = \sum_{s=1}^n a_s b_s$. Vectors $r = (r^1,\, r^2,\, \ldots,\, r^n)^T$ and $k = (k^1,\, k^2,\, \ldots,\, k^n)^T$ are the gradient vectors of real and imaginary parts of λ at $p = p_0$, given by

$$r^s + i\,k^s = \frac{v_0^T \dfrac{\partial A}{\partial p_s}\, u_0}{v_0^T u_0}, \quad s = 1,\, 2,\, \ldots,\, n. \tag{1.8}$$

There are two complex-conjugate quantities λ_1, $\overline{\lambda}_1 = (r,\, e) \pm i\,(k,\, e)$ corresponding to a complex-conjugate pair of simple eigenvalues λ_0, $\overline{\lambda}_0 = \alpha_0 \pm i\,\omega_0$. The increments of these eigenvalues are given in the form

$$\lambda,\, \overline{\lambda} = \alpha_0 + (r,\, e)\,\varepsilon \pm i\,[\,\omega_0 + (k,\, e)\,\varepsilon\,] + o(\varepsilon). \tag{1.9}$$

In the case of a real eigenvalue $\lambda_0 = \alpha_0$, the vector $k = 0$.

1.2. Double eigenvalue. Let us consider the case of a double eigenvalue λ_0 with the length of the Jordan chain equal to 2. This means that at $p = p_0$ the eigenvalue λ_0 corresponds to an eigenvector u_0 and an associated vector u_1 governed by the equations

$$\begin{aligned} A_0\, u_0 &= \lambda_0\, u_0, \\ A_0\, u_1 &= \lambda_0\, u_1 + u_0. \end{aligned} \tag{1.10}$$

For a left eigenvector v_0 and an associated vector v_1 we have

$$v_0^T A_0 = \lambda_0 v_0^T, \quad v_1^T A_0 = \lambda_0 v_1^T + v_0^T. \tag{1.11}$$

From (1.10) and (1.11) it directly follows that the vectors u_0, u_1, v_0, and v_1 are related by the conditions [16]

$$v_0^T u_0 = 0, \quad v_1^T u_0 = v_0^T u_1 \neq 0. \tag{1.12}$$

In the case of a double eigenvalue we have expansions [16, 11]

$$\lambda = \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\,\lambda_2 + \varepsilon^{3/2}\lambda_3 + \cdots, \quad u = u_0 + \varepsilon^{1/2}w_1 + \varepsilon\,w_2 + \varepsilon^{3/2}w_3 + \cdots. \tag{1.13}$$

Substituting (1.13) and (1.2) into (1.1) and using (1.10)–(1.12) we obtain expressions for determining λ_1 and λ_2:

$$\lambda_1 = \pm\sqrt{\frac{v_0^T A_1 u_0}{v_0^T u_1}}, \quad \lambda_2 = \frac{v_0^T A_1 u_1 + v_1^T A_1 u_0 - \lambda_1^2\, v_1^T u_1}{2\, v_0^T u_1}. \tag{1.14}$$

Expressions (1.13), (1.14) are correct if $v_0^T A_1 u_0 \neq 0$ (the condition Γ in [16]).

Note that the vectors u_0 and v_0 are defined up to arbitrary nonzero multipliers; the vectors u_1 and v_1 are defined up to additive terms αu_0 and βv_0, respectively, where α and β are arbitrary constants. However, the values of λ_1 and λ_2 in (1.14) don't depend on the way that the vectors u_0, u_1, v_0, and v_1 are chosen.

Assuming that the vectors u_0, u_1 are fixed we use the following normalization conditions for v_0 and v_1:

$$v_0^T u_1 = 1, \quad v_1^T u_1 = 0. \tag{1.15}$$

Combining the expressions (1.14) with (1.3), (1.13) and the normalization conditions (1.15) gives

$$\lambda = \lambda_0 \pm \sqrt{[\,(f_1,\, e) + i\,(q_1,\, e)\,]\,\varepsilon} + \frac{1}{2}\,[\,(f_2,\, e) + i\,(q_2,\, e)\,]\,\varepsilon + o(\varepsilon), \tag{1.16}$$

where components of the vectors $f_j = (f_j^1,\, f_j^2,\, \ldots,\, f_j^n)$, $q_j = (q_j^1,\, q_j^2,\, \ldots,\, q_j^n)$, $j = 1,\, 2$, are real and imaginary parts of quantities defined by

$$f_1^s + i\,q_1^s = v_0^T \frac{\partial A}{\partial p_s}\, u_0, \quad f_2^s + i\,q_2^s = v_0^T \frac{\partial A}{\partial p_s}\, u_1 + v_1^T \frac{\partial A}{\partial p_s}\, u_0, \quad s = 1,\, 2,\, \ldots,\, n. \tag{1.17}$$

If λ_0 is a real number, then the vectors $q_1 = q_2 = 0$.

1.3. Triple eigenvalue. Consider the case of a triple eigenvalue that is characterized by a Jordan chain of the length 3. This means that there are an eigenvector u_0 and associated vectors u_1, u_2 satisfying the equations

$$\begin{aligned} A_0\, u_0 &= \lambda_0\, u_0, \\ A_0\, u_1 &= \lambda_0\, u_1 + u_0, \\ A_0\, u_2 &= \lambda_0\, u_2 + u_1. \end{aligned} \tag{1.18}$$

For a left eigenvector v_0 and associated vectors v_1 and v_2 we have

$$\begin{aligned} v_0^T A_0 &= \lambda_0\, v_0^T, \\ v_1^T A_0 &= \lambda_0\, v_1^T + v_0^T, \\ v_2^T A_0 &= \lambda_0\, v_2^T + v_1^T. \end{aligned} \tag{1.19}$$

The vectors u_j, v_j, $j = 1, 2, 3$, are related by the conditions

$$\begin{aligned} v_0^T u_0 = v_0^T u_1 &= v_1^T u_0 = 0, \\ v_0^T u_2 = v_1^T u_1 &= v_2^T u_0 \neq 0, \\ v_1^T u_2 &= v_2^T u_1. \end{aligned} \tag{1.20}$$

These conditions can be proved by means of (1.18), (1.19). Assuming that the vectors u_j, $j = 1, 2, 3$, are fixed we use the following normalization conditions for the vectors v_j, $j = 1, 2, 3$:

$$v_0^T u_2 = 1, \quad v_1^T u_2 = v_2^T u_2 = 0. \tag{1.21}$$

These conditions define the vectors v_j, $j = 1, 2, 3$, uniquely.

The eigenvalue λ_0 generally splits to three simple eigenvalues due to perturbation of parameters. Then eigenvalues and eigenvectors can be given in the form [16, 11]

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon^{1/3}\lambda_1 + \varepsilon^{2/3}\lambda_2 + \varepsilon\,\lambda_3 + +\varepsilon^{4/3}\lambda_4 + \cdots, \\ u &= u_0 + \varepsilon^{1/3} w_1 + \varepsilon^{2/3} w_2 + \varepsilon\, w_3 + \varepsilon^{4/3} w_4 + \cdots. \end{aligned} \tag{1.22}$$

Substituting expansions (1.22) and (1.2) into (1.1) and using (1.18)–(1.21) we obtain expressions for the first three coefficients λ_j, $j = 1, 2, 3$:

$$\begin{aligned} \lambda_1 &= \sqrt[3]{v_0^T A_1 u_0}, \\ \lambda_2 &= \frac{v_0^T A_1 u_1 + v_1^T A_1 u_0}{3\,\lambda_1}, \\ \lambda_3 &= \frac{1}{3}\left(v_0^T A_1 u_2 + v_1^T A_1 u_1 + v_2^T A_1 u_0\right). \end{aligned} \tag{1.23}$$

These expressions are correct if $v_0^T A_1 u_0 \neq 0$ (the condition Γ in [16]). In this case the first expression of (1.23) defines three different complex roots λ_1. Then values of λ_2 and λ_3 are determined for each root λ_1 from the second and third expressions of (1.23).

Combining expressions (1.23) with (1.22) and (1.3) gives

$$\begin{aligned} \lambda = \lambda_0 &+ \sqrt[3]{[\,(h_1,\, e) + i\,(t_1,\, e)\,]\,\varepsilon} \\ &+ \frac{(h_2,\, e) + i\,(t_2,\, e)}{3\sqrt[3]{(h_1,\, e) + i\,(t_1,\, e)}}\,\varepsilon^{2/3} + \frac{1}{3}\,[\,(h_3,\, e) + i\,(t_3,\, e)\,]\,\varepsilon + o(\varepsilon), \end{aligned} \tag{1.24}$$

where components of the vectors $h_j = (h_j^1,\, h_j^2,\, \dots,\, h_j^n)$, $t_j = (t_j^1,\, t_j^2,\, \dots,\, t_j^n)$, $j = 1, 2, 3$, are real and imaginary parts of the quantities

$$\begin{aligned} h_1^s + i\, t_1^s &= v_0^T \frac{\partial A}{\partial p_s}\, u_0, \\ h_2^s + i\, t_2^s &= v_0^T \frac{\partial A}{\partial p_s}\, u_1 + v_1^T \frac{\partial A}{\partial p_s}\, u_0, \\ h_3^s + i\, t_3^s &= v_0^T \frac{\partial A}{\partial p_s}\, u_2 + v_1^T \frac{\partial A}{\partial p_s}\, u_1 + v_2^T \frac{\partial A}{\partial p_s}\, u_0, \\ s &= 1, 2, \dots, n. \end{aligned} \tag{1.25}$$

The cubic roots in the second and third terms of the right-hand side of (1.24) are the same and take three different complex values.

If λ_0 is a real number, then the vectors $t_j = 0$, $j = 1, 2, 3$.

2. One- and two-parameter families of matrices. Let us consider a linear evolutionary system of the form

$$\dot{y} = A\, y, \tag{2.1}$$

where A is a real autonomous $m \times m$ matrix and y is a real vector of dimension m. The system is stable (asymptotically stable) if all eigenvalues λ of the matrix A have negative real parts, $\operatorname{Re}\lambda < 0$. If there exists at least one eigenvalue such that $\operatorname{Re}\lambda > 0$, the system is unstable. If there are some eigenvalues with $\operatorname{Re}\lambda = 0$ while for all others $\operatorname{Re}\lambda < 0$, we have a boundary point.

A *family* of matrices is a mapping $A : \Lambda \longrightarrow \mathbf{R}^{m^2}$ of the parameter space into the space of matrices. The set of values $p \in \Lambda$, such that $A(p)$ is a stable matrix, is called the stability domain. First, let us consider a one-parameter family $A(p)$, $p \in \mathbf{R}$. The stability domain boundary of a *generic* one-parameter family is characterized by one simple eigenvalue $\lambda = 0$ or by one pair of complex-conjugate simple eigenvalues $\lambda = \pm i\,\omega$ of the matrix A [3, 4]. In the technical literature these cases are called divergence and flutter boundaries, respectively.

Using (1.9) we have

$$\operatorname{Re}\lambda = r\,(p - p_0) + o(p - p_0) \tag{2.2}$$

for a simple eigenvalue λ in the neighborhood of the stability boundary point p_0, $\operatorname{Re}\lambda_0 = 0$. Hence, location of the stability and instability domains is determined by the sign of the quantity

$$r = \operatorname{Re}\left[\left(v_0^T\, dA/dp\, u_0\right)/\left(v_0^T u_0\right)\right].$$

For example, if $r > 0$, then the system is stable (Re $\lambda < 0$) at $p < p_0$ and unstable (Re $\lambda > 0$) at $p > p_0$ for p sufficiently close to p_0. Note that $r \neq 0$ in the generic case, i.e., the case $r = 0$ can be removed by an arbitrarily small shift of the family.

In the case of a two-parameter generic family $A(p)$, $p \in \mathbf{R}^2$, the stability boundary is a smooth curve whose only singularities are corners. The curve in nonsingular points is characterized by a simple eigenvalue $\lambda = 0$ or a simple pair $\lambda = \pm i\,\omega$ and has a normal vector r defined in (1.8). From (1.9) it follows that the normal r lies in the instability domain. The corners correspond to matrices $A(p)$ of the three following types (strata) [3, 4, 5]: $F_1(0^2)$—double eigenvalue $\lambda = 0$ with the

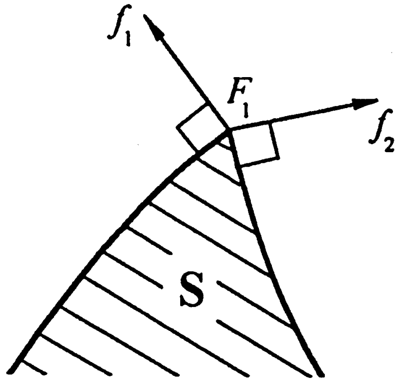

FIG. 2.1. *A corner of the stability domain.*

corresponding Jordan block of the order 2; $F_2(0, \pm i\,\omega)$— simple eigenvalues $0, \pm i\,\omega$; $F_3(\pm i\,\omega_1, \pm i\,\omega_2)$—two pairs of simple different eigenvalues $\pm i\,\omega_1, \pm i\,\omega_2$. Other types of singularities can be destroyed by an arbitrarily small shift of the family.

Using expansions (1.16) for a double eigenvalue $\lambda_0 = 0$ we have

$$\lambda = \pm\sqrt{(f_1, e)\,\varepsilon} + \frac{1}{2}(f_2, e)\,\varepsilon + o(\varepsilon), \tag{2.3}$$

where vectors f_1 and f_2 are defined by means of (1.17) and e is a vector of variation (direction). In the generic case the vectors f_1 and f_2 are linearly independent. For an arbitrary fixed vector e, such that $(f_1, e) < 0$ and $(f_2, e) < 0$, we have $\operatorname{Re}\lambda < 0$ (stability) for sufficiently small ε. If at least one of these inequalities has the opposite sign, we have $\operatorname{Re}\lambda > 0$ (instability).

For the following presentation we need a concept of *tangent cone*. A tangent cone to the stability domain at the boundary point is a set of direction vectors of the curves starting at this point and lying in the stability domain [9]. A tangent cone is nondegenerate if it cuts out on a sphere a set of nonzero measure. Otherwise, the cone is called degenerate. A tangent cone can be considered as a linear approximation of the stability domain.

In accordance with (2.3) a tangent cone at the boundary point corresponding to the stratum $F_1(0^2)$ takes the form

$$K_{F_1} = \big\{\, e : (f_1, e) \le 0,\ \ (f_2, e) \le 0 \,\big\}. \tag{2.4}$$

Using expression (2.2) for a simple eigenvalue we similarly obtain tangent cones at boundary points corresponding to the strata $F_2(0, \pm i\,\omega)$ and $F_3(\pm i\,\omega_1, \pm i\,\omega_2)$:

$$K_{F_2} = \big\{\, e : (r_0, e) \le 0,\ \ (r, e) \le 0 \,\big\}, \tag{2.5}$$

$$K_{F_3} = \big\{\, e : (r_1, e) \le 0,\ \ (r_2, e) \le 0 \,\big\}. \tag{2.6}$$

Here the vectors r_0, r, r_1, r_2 correspond to the simple eigenvalues 0, $\pm i\,\omega$, $\pm i\,\omega_1$, $\pm i\,\omega_2$, respectively, and are calculated using (1.8). In the generic case the vectors r, r_0 and also r_1, r_2 are linearly independent.

Using the relations (2.4)–(2.6) for tangent cones we can find tangent vectors to the stability boundary. For example, in the case of a singular point of the type F_1

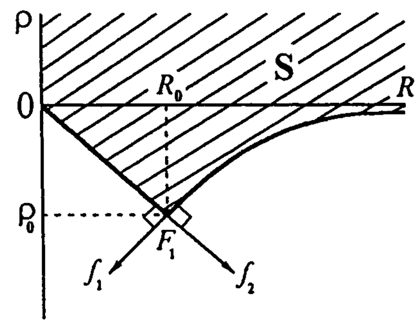

FIG. 3.1. *The stability domain of equilibrium of the voltaic arc in electric circuit.*

tangent vectors g_1 and g_2 can be found from the following system of linear equations (see Figure 2.1):

$$(f_i,\, g_j) = -\delta_{ij}, \quad i,\, j = 1,\, 2, \tag{2.7}$$

where δ_{ij} is Kronecker's delta and f_1, f_2 are the vectors from (2.4).

The inequalities in (2.4)–(2.6) define an intersection of two halfplanes. Hence, the stability domain wedges into the instability domain with the angle of wedge less than π; see Figure 2.1. This is a quantitative justification of the Arnold's principle of "fragility of all good things" [3, 4, 5] and quasi convexity of the stability domain [9].

3. Example: Stability of equilibrium of a voltaic arc in electric circuit. As an example let us consider a stability problem of equilibrium of a circuit consisting of a voltaic arc, resister R, inductance L, and shunting capacitor C connected in series. Linearized differential equations near the equilibrium of the system have the form [1]

$$\begin{aligned} \frac{d\xi}{dt} &= -\frac{\rho\xi}{L} + \frac{\eta}{L}, \\ \frac{d\eta}{dt} &= -\frac{\xi}{C} - \frac{\eta}{RC}, \end{aligned} \tag{3.1}$$

where $\xi(t)$, $\eta(t)$ are, respectively, an electric current and a voltage in the voltaic arc, and ρ is a resistance of the arc.

The system (3.1) depends on four parameters: three positive quantities L, C, R and parameter ρ, which can take both positive and negative values. Assuming that the parameters L and C are fixed, we consider the stability problem on the plane of two parameters: $p_1 = R$ and $p_2 = \rho$. The matrix A corresponding to the system (3.1) is

$$A = \begin{pmatrix} -\dfrac{\rho}{L} & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{CR} \end{pmatrix}. \tag{3.2}$$

The characteristic equation of the system takes the form

$$\lambda^2 + \left(\frac{1}{RC} + \frac{\rho}{L}\right)\lambda + \frac{1}{LC}\left(\frac{\rho}{R} + 1\right) = 0. \tag{3.3}$$

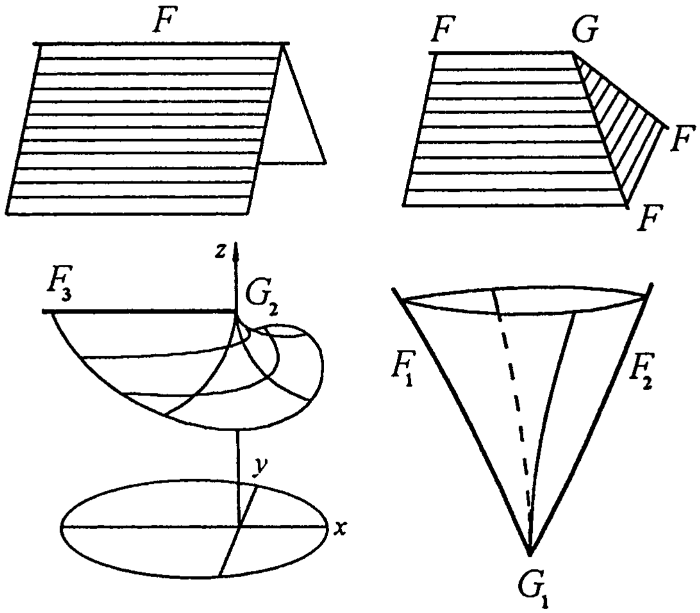

FIG. 4.1. *Singularities of the stability boundary of a three-parameter generic family.*

At the point $R_0 = \sqrt{L/C},\ \ \rho_0 = -\sqrt{L/C}$ the characteristic equation (3.3) has a double root $\lambda_0 = 0$ with the length of the Jordan chain equal to 2. The equations for Jordan chains (1.10), (1.11) yield at this point

$$u_0 = \begin{pmatrix} 1 \\ -\sqrt{L/C} \end{pmatrix}, \quad u_1 = \begin{pmatrix} 0 \\ L \end{pmatrix}, \quad v_0 = \begin{pmatrix} 1/\sqrt{LC} \\ 1/L \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using these vectors and the matrix A from (3.2) we calculate the vectors f_1 and f_2 according to (1.17):

$$f_1 = -\frac{1}{L\sqrt{LC}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad f_2 = \frac{1}{L}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.4}$$

Thus, we have found the tangent cone (2.4) to the stability domain at the point $R = R_0,\ \ \rho = \rho_0$; see Figure 3.1. The tangent vectors to the stability boundary (2.7) up to a positive factor are $g_1 = (1, 1)^T,\ \ g_2 = (-1, 1)^T$. Hence, the angle of the wedge of the stability domain is equal to $\pi/2$. This result is in accordance with [1], where it has been shown that the stability boundary consists of the line $\rho = -R$, $0 \le R \le \sqrt{L/C}$ and the hyperbola $\rho = -L/(CR),\ \ \sqrt{L/C} \le R$; see Figure 3.1.

4. Three-parameter family of matrices. Consider a generic three-parameter family of matrices $A(p)$, $p \in \mathbf{R}^3$. The stability domain boundary of the family is a smooth surface characterized by one simple eigenvalue $\lambda = 0$ or a pair of simple eigenvalues $\lambda = \pm i\,\omega$ [3, 4]. The normal vector r to this surface is defined by the relation (1.8) in the same way as in the two-parameter case. The vector r lies in the instability domain. According to [3, 4] the only singularities of the stability boundary of a generic three-parameter family are of four types: dihedral angle, trihedral angle, deadlock of an edge, and break of an edge; see Figure 4.1.

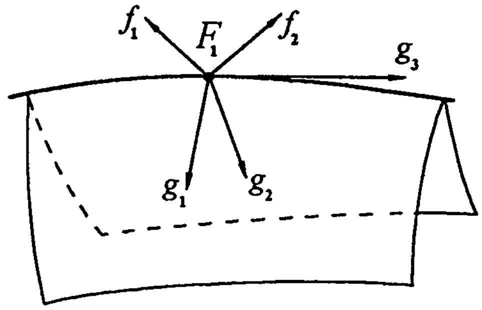

FIG. 4.2. *An edge of the stability domain.*

The dihedral angle singularity is connected with the strata $F_1(0^2)$, $F_2(0, \pm i\,\omega)$, $F_3(\pm i\,\omega_1, \pm i\,\omega_2)$ examined in section 2. Therefore tangent cones to the stability domain at these singular points are determined by the relations (2.4)–(2.6), respectively.

The tangent cone K_{F_1} can be written in a more suitable form introducing vectors g_1, g_2, g_3 by the equations

$$(f_i,\, g_j) = -\delta_{ij}, \quad i = 1, 2, \quad j = 1, 2, 3, \tag{4.1}$$

where f_1, f_2 are the vectors from (2.4). The equations (4.1) are solvable because the vectors f_1 and f_2 are linearly independent for a generic family. The vector g_3 is directed along the edge and the vectors g_1, g_2 are tangent to the sides of the dihedral angle; see Figure 4.2. Using these vectors the set (2.4) can be written in the form

$$K_{F_1} = \big\{\, e : \ e = \alpha\, g_1 + \beta\, g_2 + \gamma\, g_3; \ \ \alpha, \beta, \gamma \in \mathbf{R},\ \alpha \ge 0,\ \beta \ge 0 \,\big\}. \tag{4.2}$$

Substituting the expression $e = \alpha\, g_1 + \beta\, g_2 + \gamma\, g_3$ into (2.4) and using (4.1) we find $(f_1, e) = -\alpha \le 0$, $(f_2, e) = -\beta \le 0$. It proves the representation (4.2). Similar representations can be obtained for the tangent cones K_{F_2} and K_{F_3} using in (4.1) the vectors r_0, r and r_1, r_2 instead of f_1, f_2, respectively.

The trihedral angle singularity is characterized by the following strata [3, 4]: $G_3(0^2, \pm i\,\omega)$—a double eigenvalue $\lambda = 0$ with the length of the Jordan chain equal to 2 and a pair of simple pure imaginary eigenvalues; $G_4(0, \pm i\,\omega_1, \pm i\,\omega_2)$—simple $\lambda = 0$ and two different pairs of simple pure imaginary eigenvalues; $G_5(\pm i\,\omega_1, \pm i\,\omega_2, \pm i\,\omega_3)$—three different pairs of simple pure imaginary eigenvalues.

Note that these strata differ from the strata F_1, F_2, F_3 by the presence of an additional pair of simple eigenvalues of the type $\lambda = \pm i\,\omega$. Therefore tangent cones at the boundary points of these types, similarly to (2.4)–(2.6), can be written in the form

$$K_{G_3} = \big\{\, e : (f_1, e) \le 0,\ (f_2, e) \le 0,\ (r, e) \le 0 \,\big\}, \tag{4.3}$$

$$K_{G_4} = \big\{\, e : (r_0, e) \le 0,\ (r_1, e) \le 0,\ (r_2, e) \le 0 \,\big\}, \tag{4.4}$$

$$K_{G_5} = \big\{\, e : (r_1, e) \le 0,\ (r_2, e) \le 0,\ (r_3, e) \le 0 \,\big\}, \tag{4.5}$$

where the vectors r_0, r, r_j, $j = 1, 2, 3$ correspond to the eigenvalues 0, $\pm i\,\omega$, $\pm i\,\omega_j$, $j = 1, 2, 3$, respectively, and are defined in (1.8). The vectors f_1 and f_2 correspond

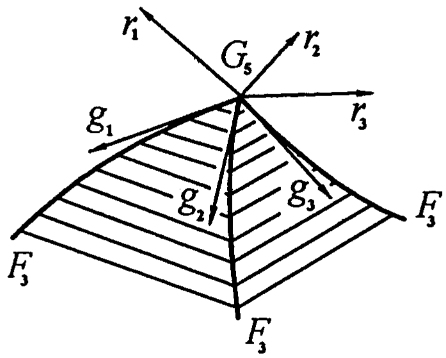

FIG. 4.3. *The singularity trihedral angle.*

to the double eigenvalue $\lambda = 0$ and are defined in (1.17). The sets (4.3)–(4.5) describe a trihedral angle, which is located entirely in a closed half-space; see Figure 4.3.

As in (4.1), using the vectors determining the cones, we can find vectors g_1, g_2, g_3 tangential to the edges of the trihedral angle; see Figure 4.3. For example, for the singular point of the type G_5 we have

$$(r_i,\, g_j) = -\delta_{ij}, \quad i,\, j = 1,\, 2,\, 3, \tag{4.6}$$

where r_j, $j = 1, 2, 3$ are the vectors from (4.5). With the use of these vectors the tangent cone K_{G_5} can be described in the following way:

$$K_{G_5} = \big\{\, e:\ e = \alpha\, g_1 + \beta\, g_2 + \gamma\, g_3;\ \ \alpha,\, \beta,\, \gamma \geq 0 \,\big\}. \tag{4.7}$$

Similar representations can be deduced for the cones K_{G_3} and K_{G_4}.

Note that the vectors determining dihedral and trihedral angles are linearly independent for a generic family.

Singularity deadlock of an edge is characterized by the stratum $G_2\big((\pm i\,\omega)^2\big)$—a pair of double pure imaginary eigenvalues $\lambda = \pm i\,\omega$ with the length of the Jordan chain equal to 2. It is well known that the stability domain in the neighborhood of this singularity up to a smooth change of parameters (diffeomorphism) is given by [3, 4]

$$z + \Big|\, \mathrm{Re}\sqrt{x + i\,y}\, \Big| < 0. \tag{4.8}$$

The stability boundary of (4.8) is a half of the so-called Witney–Cayley umbrella surface [4, 5]. The tangent cone to the domain (4.8) at the singular point G_2, i.e., at $x = y = z = 0$, is degenerate and represents a plane angle

$$K^0_{G_2} = \big\{\, e = (e_1,\, e_2,\, e_3):\ e_1 \leq 0,\ \ e_2 = 0,\ \ e_3 \leq 0 \,\big\}.$$

Note that the singularity G_2 is formed by a collision of two different simple pure imaginary eigenvalues $i\,\omega_1$ and $i\,\omega_2$ at the singular point when they move along the edge of the type F_3.

Let us calculate the tangent cone for G_2 in the generic case. Using the expansions (1.16) for a double eigenvalue $\lambda = i\,\omega$ we have

$$\lambda = i\,\omega \pm \sqrt{[\,(f_1,\, e) + i\,(q_1,\, e)\,]\,\varepsilon} + \frac{1}{2}\,[\,(f_2,\, e) + i\,(q_2,\, e)\,]\,\varepsilon + o(\varepsilon), \tag{4.9}$$

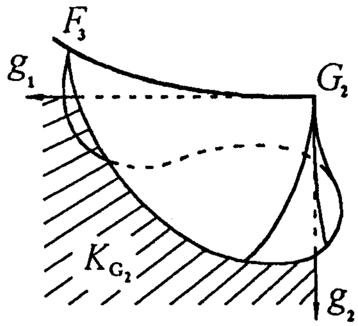

FIG. 4.4. *The singularity deadlock of an edge.*

where the vectors $f_1,\ q_1,\ f_2,\ q_2$ correspond to $\lambda = i\,\omega$ and are defined in (1.17). The expansions for a complex-conjugate double eigenvalue $\overline{\lambda} = -i\,\omega$ can be found by taking complex conjugation of (4.9).

If either $(f_1, e) > 0$ or $(q_1, e) \neq 0$ under the radical in (4.9), then one of the eigenvalues has positive real part for sufficiently small ε (instability). In the case when $(f_1, e) < 0,\ (q_1, e) = 0$, the second (with the radical) term is a pure imaginary number. Hence, for $(f_2, e) < 0$ and small ε we have $\operatorname{Re}\lambda < 0$ (stability) and for $(f_2, e) > 0$ we have $\operatorname{Re}\lambda > 0$ (instability). Therefore, the tangent cone to the stability domain at a singular point G_2 is the plane angle of the form

$$K_{G_2} = \big\{\, e : (f_1, e) \le 0,\ \ (f_2, e) \le 0,\ \ (q_1, e) = 0 \,\big\}. \tag{4.10}$$

From (4.9) it directly follows that all smooth curves, emitted from the singular point along the direction e, satisfying the conditions $(f_1, e) < 0,\ (f_2, e) < 0,\ (q_1, e) = 0$, lie in the stability domain for rather small ε.

Determining the vectors g_1, g_2 by

$$(f_i, g_j) = -\delta_{ij}, \quad (q_1, g_j) = 0, \quad i, j = 1, 2,$$

we can write (4.10) in the form

$$K_{G_2} = \big\{\, e : \ e = \alpha\, g_1 + \beta\, g_2;\ \ \alpha, \beta \ge 0 \,\big\}. \tag{4.11}$$

The vectors g_1 and g_2 are directed along the sides of the plane angle K_{G_2}. The vector g_1 is tangent to the edge F_3 of the stability domain characterized by two different pairs of simple eigenvalues $\pm i\,\omega_1,\ \pm i\,\omega_2$; see Figure 4.4.

Note, that the vectors $f_1,\ f_2,\ q_1$ are linearly independent for a generic family of matrices.

5. Singularity break of an edge. The singularity break of an edge is characterized by the stratum $G_1(0^3)$—by one triple zero eigenvalue of the matrix $A(p_0)$ with the Jordan chain of the length equal to 3. The expansion of a triple eigenvalue is described by (1.24). The cubic root in (1.24) takes three different complex values. This means that if $(h_1, e) \neq 0$ (note that $t_1 = 0$ since $\lambda_0 = 0$), then at least one eigenvalue has positive real part (instability). Hence, the tangent cone to the stability domain lies in the plane $(h_1, e) = 0$, where the expansion (1.24) is not valid due to violation of the Γ condition $v_0^T A_1 u_0 = (h_1, e) \neq 0$. Therefore the tangent cone in this case cannot be found with the method used for the investigation of the previous singularities. For this reason we take another approach to study this singularity.

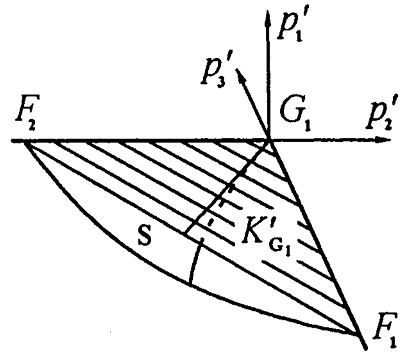

FIG. 5.1. *The singularity break of an edge.*

Let us construct a versal deformation of the matrix $A_0 = A(p_0)$ that is a smooth matrix family $A'(p')$, $p' \in \mathbf{R}^d$, such that any smooth family $A(p)$, $A(p_0) = A_0$, can be represented in a neighborhood of $p = p_0$ in the form

$$A(p) = C(p)A'(\varphi(p))C^{-1}(p), \tag{5.1}$$

where C(p) is a smooth family of nonsingular matrices, $p' = \varphi(p)$ is a smooth mapping from a neighborhood of the point p_0 in $\mathbf{R}^3$ into a neighborhood of the origin of coordinate system in $\mathbf{R}^d$, and $\varphi_1(p_0) = \varphi_2(p_0) = \cdots = \varphi_d(p_0) = 0$. A versal deformation with the minimum possible number of parameters d is called miniversal one. The miniversal deformation of the matrix A_0 can be chosen in the block diagonal form [2, 7]

$$A'(p') = A'(0) + B(p'). \tag{5.2}$$

Here $A'(0)$ is the Jordan form of A_0 and $B(p')$ is a family of block diagonal matrices whose blocks are determined in accordance with the structure of $A'(0)$. The first block of $A'(p')$ corresponding to the triple zero eigenvalue (0^3) can be taken in the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_1' & p_2' & p_3' \end{pmatrix}. \tag{5.3}$$

The other blocks correspond to eigenvalues with negative real parts. Due to (5.1) the characteristic equations for the matrices $A(p)$ and $A'(p')$, $p' = \varphi(p)$, coincide identically. Stability of the matrix $A'(p')$ in a neighborhood of the point $p' = 0$ is determined by the first block (5.3) due to its block diagonal structure. The characteristic equation of (5.3) takes the form $\lambda^3 - p_3'\lambda^2 - p_2'\lambda - p_1' = 0$. The stability domain of this equation is found using the Routh–Hurwitz conditions

$$\begin{gathered} p_1' + p_2'p_3' > 0, \\ p_1' < 0, \quad p_2' < 0, \quad p_3' < 0. \end{gathered} \tag{5.4}$$

This domain in 3-parameter space p_1', p_2', p_3' is shown in Figure 5.1. Directly from (5.4) we find that the tangent cone to the stability domain at $p' = 0$ is degenerate. It is defined by the relations

$$e_1' = 0, \quad e_2' \le 0, \quad e_3' \le 0. \tag{5.5}$$

Let us calculate the vectors h'_j, $j = 1, 2, 3$, defining collapse of a triple zero eigenvalue of the matrix $A'(0)$. Finding eigenvectors and associated vectors u'_j, v'_j, $j = 1, 2, 3$, satisfying the normalization conditions (1.21), and using (1.25) we obtain

$$\begin{aligned} h'_1 &= (1, 0, 0, 0, \ldots, 0)^T, \\ h'_2 &= (0, 1, 0, 0, \ldots, 0)^T, \\ h'_3 &= (0, 0, 1, 0, \ldots, 0)^T. \end{aligned}$$

By means of these vectors the tangent cone (5.5) can be given in the form

$$K'_{G_1} = \left\{ e' \in \mathbf{R}^d \ : (h'_1, e') = 0,\ (h'_2, e') \le 0,\ (h'_3, e') \le 0 \right\}. \tag{5.6}$$

Let us determine a tangent cone for the family $A(p)$. For this purpose we have to find a relation between the vectors h'_j and h_j, $j = 1, 2, 3$. Let u'_j, v'_j, $j = 0, 1, 2$, be left and right eigenvectors and associated vectors of the matrix $A'(0)$, corresponding to the triple eigenvalue $\lambda_0 = 0$ and satisfying the normalization conditions (1.21). Then, using (5.1) we get relations between u'_j, v'_j, $j = 1, 2, 3$, and eigenvectors and associated vectors u_j, v_j, $j = 1, 2, 3$, of the matrix A_0

$$\begin{aligned} u_j &= C(p_0) u'_j, \\ v_j^T &= {v'_j}^T C^{-1}(p_0)\ , \quad j = 0, 1, 2. \end{aligned} \tag{5.7}$$

We differentiate the expression (5.1) with respect to p_j and find the value of the derivative at $p = p_0$, $p' = \varphi(p_0) = 0$,

$$\frac{\partial A}{\partial p_j} = \frac{\partial C}{\partial p_j} A' C^{-1} + C A' \frac{\partial C^{-1}}{\partial p_j} + \sum_{s=1}^{d} \left(C \frac{\partial A'}{\partial p'_s} C^{-1} \right) \frac{\partial \varphi_s}{\partial p_j}\ , \quad j = 1, 2, 3. \tag{5.8}$$

Multiplying (5.8) by v_0^T and u_0 from left and right, respectively, we have

$$\begin{aligned} h_1^j &= v_0^T \frac{\partial A}{\partial p_j} u_0 = v_0^T \frac{\partial C}{\partial p_j} A' C^{-1} u_0 \\ &+ v_0^T C A' \frac{\partial C^{-1}}{\partial p_j} u_0 + \sum_{s=1}^{d} v_0^T C \frac{\partial A'}{\partial p'_s} \frac{\partial \varphi_s}{\partial p_j} C^{-1} u_0 \\ &= \sum_{s=1}^{d} \frac{\partial \varphi_s}{\partial p_j} \left[{v'_0}^T \frac{\partial A'}{\partial p'_s} u'_0 \right], \quad j = 1, 2, 3. \end{aligned} \tag{5.9}$$

Here we have used the relations (5.7) and $A'(0) u'_0 = 0$, ${v'_0}^T A'(0) = 0$. Thus, from (5.9) we get the relation between the vectors h_1 and h'_1:

$$h_1^T = {h'_1}^T D_\varphi, \quad D_\varphi = \left[\frac{\partial \varphi_i}{\partial p_j} \right], \quad i = 1, 2, \ldots, d, \quad j = 1, 2, 3.$$

Analogously, we can prove this relation for the vectors h_2, h'_2 and h_3, h'_3. In this proof expressions (5.7), (5.8), and the equalities

$$\begin{gathered} v_s^T \frac{\partial C}{\partial p_i} C^{-1} u_j + v_s^T C \frac{\partial C^{-1}}{\partial p_i} u_j = v_s^T \frac{\partial C C^{-1}}{\partial p_i} u_j = 0, \\ i = 1, 2, \ldots, n, \quad s, j = 0, 1, \end{gathered}$$

are used. Thus, the vectors h_s and h'_s are related by

$$h_s^T = {h'_s}^T D_\varphi, \quad s = 1,2,3. \tag{5.10}$$

Now we find a relation between direction vectors e and e' from the tangent cones in $\mathbf{R}^3$ and $\mathbf{R}^d$, respectively:

$$e'_i = \frac{\partial p'_i}{d\varepsilon} = \sum_{j=1}^{d} \frac{\partial \varphi_i}{\partial p_j}\frac{dp_j}{d\varepsilon} = \sum_{j=1}^{d} \frac{\partial \varphi_i}{\partial p_j} e_j, \quad i = 1,2,\dots,d, \quad j = 1,2,3.$$

Consequently,

$$e' = D_\varphi e. \tag{5.11}$$

Any curve $p(\varepsilon)$, $p(0) = p_0$, with the direction $e = dp/d\varepsilon$, lying in the stability domain, corresponds to a curve $p'(\varepsilon) = \varphi(p(\varepsilon))$ with the direction $e' = D_\varphi e$ lying in the stability domain in $\mathbf{R}^d$. Similarly, for any curve $p'(\varepsilon)$, $p'(0) = 0$, $dp'/d\varepsilon = e'$, lying in the stability domain in $\mathbf{R}^d$, in the case of linearly independent vectors h_s, $s = 1,2,3$, there exist curves $p(\varepsilon)$, $p(0) = p_0$, with directions e, related to e' by (5.11), and lying in the stability domain. In the case of the generic family of matrices $A(p)$ the vectors h_s, $s = 1,2,3$, are linearly independent.

Multiplying (5.10) by e and using (5.11) we obtain

$$h_s^T e = {h'_s}^T e', \quad s = 1,2,3. \tag{5.12}$$

Using (5.6) and (5.12) we find the tangent cone to the stability domain at the singular point $G_1(0^3)$ in the form

$$K_{G_1} = \big\{\, e : (h_1, e) = 0,\ (h_2, e) \le 0,\ (h_3, e) \le 0 \,\big\}. \tag{5.13}$$

The tangent cone K_{G_1} is degenerate and represents a plane angle. Recall that the vectors h_s, $s = 1,2,3$, determining the cone, are defined by (1.25) and need only eigenvectors and associated vectors corresponding to the triple zero eigenvalue and the derivatives of A with respect to p_j, $j = 1,2,3$, at the point under consideration.

Introducing the vectors g_1, g_2 by formulae

$$(g_j, h_{4-s}) = -\delta_{js}, \quad j = 1,2, \quad s = 1,2,3,$$

the tangent cone K_{G_1} can be written in the form

$$K_{G_1} = \big\{\, e : e = \alpha g_1 + \beta g_2, \quad \alpha, \beta \ge 0 \,\big\},$$

where the vectors g_1, g_2 are tangent to the edges of the singularity.

6. A simple model of a damped follower force column. A simple, two-degrees-of-freedom pendulum loaded by a follower force P has been studied by Ziegler [17] and in the present extended version with two different damping parameters by Herrmann and Jong [8]. Boundary surface of the stability domain of this system was plotted and studied by Seyranian and Pedersen [15]. We consider this example from the point of view of singularities of the stability boundary and show that the effects known as destabilization due to damping [8] and uncertainty of the critical load when damping parameters tend to zero [14, 15] are closely related to the deadlock of an edge singularity, which takes place at the point of the critical load of the system with no damping, and the dihedral angle singularity at lower values of the load.

The linearized equations of free motion of the pendulum in nondimensional variables are [8]

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \ddot\varphi_1 \\ \ddot\varphi_2 \end{pmatrix} + \begin{pmatrix} \gamma_1+\gamma_2 & -\gamma_2 \\ -\gamma_2 & \gamma_2 \end{pmatrix}\begin{pmatrix} \dot\varphi_1 \\ \dot\varphi_2 \end{pmatrix} + \begin{pmatrix} 2-p & -1+p \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{6.1}$$

where γ_1 and γ_2 are independent nonnegative damping parameters and p is a magnitude of the follower force.

Introducing variables $\varphi_3 = \dot\varphi_1$ and $\varphi_4 = \dot\varphi_2$, (6.1) takes the form

$$\dot\varphi = A\varphi, \qquad \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T, \tag{6.2}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p/2-3/2 & 1-p/2 & -\gamma_1/2-\gamma_2 & \gamma_2 \\ 5/2-p/2 & p/2-2 & \gamma_1/2+2\gamma_2 & -2\gamma_2 \end{pmatrix}. \tag{6.3}$$

We investigate the stability domain of the system in the space of three parameters (γ_1, γ_2, p). The characteristic equation of the system (6.2), (6.3) is

$$2\lambda^4 + (\gamma_1 + 6\gamma_2)\lambda^3 + (7 - 2p + \gamma_1\gamma_2)\lambda^2 + (\gamma_1 + \gamma_2)\lambda + 1 = 0. \tag{6.4}$$

At $\gamma_1 = \gamma_2 = 0$ (the system without damping) we find

$$\lambda^2 = \frac{1}{2}\left(p - 7/2 \pm \sqrt{(p-7/2)^2 - 2}\right). \tag{6.5}$$

Hence at $p \in [0,\, 7/2 - \sqrt{2})$ we have two different pairs of simple complex conjugate imaginary eigenvalues corresponding to the dihedral angle singularity (F_3). At $p_0 = 7/2 - \sqrt{2}$ there exists a pair of double complex conjugate imaginary eigenvalues with the Jordan chain (1.10), corresponding to the singularity deadlock of an edge (G_2). Thus, the segment $\gamma_1 = \gamma_2 = 0,\ p \in [0,\, p_0]$ is an edge of the stability boundary with the deadlock at the point $p = p_0$; see Figure 6.1.

At the point $\gamma_1 = \gamma_2 = 0,\ p \in [0,\, p_0)$ the tangent cone $K_{F_3}(p)$ has been determined in (2.6). The vectors r_1 and r_2, given by (1.8), for the matrix A from (6.3) take the form

$$r_{1,2}(p) = \frac{1}{8}\begin{pmatrix} \pm\dfrac{3/2-p}{\sqrt{(p-7/2)^2-2}} - 1 \\ \pm\dfrac{19-6p}{\sqrt{(p-7/2)^2-2}} - 6 \\ 0 \end{pmatrix}, \tag{6.6}$$

where plus corresponds to r_1 and minus to r_2. The angle between r_1 and r_2 (equal to the difference of π and the angle of the dihedron) increases with the increase of p from zero and reaches π at $p = p_0$. But at $p = p_0$ the vectors r_1 and r_2 become infinite because the radicand in (6.6) is equal to zero. Therefore the tangent cone K_{F_3}

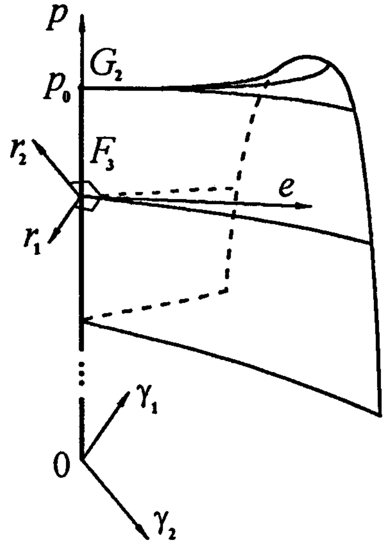

FIG. 6.1. *The stability domain of Ziegler's double pendulum.*

degenerates into the cone K_{G_2} of the deadlock of an edge singularity at $p = p_0$; see Figure 6.1.

The cone K_{G_2} has been found in (4.10), where the vectors f_1, q_1, f_2 are given by (1.17). For the matrix (6.3) they are, up to a positive factor, equal to

$$\begin{aligned} f_1 &= (0, 0, 1), \\ q_1 &= (1, -4 - 5\sqrt{2}, 0), \\ f_2 &= (-1, -6, 0). \end{aligned} \tag{6.7}$$

This cone also can be written in the form

$$K_{G_2} = \left\{ (e_1, e_2, e_3) : \ e_1 = (4 + 5\sqrt{2})e_2, \ \ e_2 \geq 0, \ \ e_3 \leq 0 \right\}. \tag{6.8}$$

In the parameter space (γ_1, γ_2, p) it represents a plane angle.

At fixed values of damping parameters γ_1, γ_2 a critical load p_{cr} is defined as the smallest value of p at which the system becomes unstable. Consider damping in the form $\gamma_1 = e_1\varepsilon$, $\gamma_2 = e_2\varepsilon$, where ε is a small positive number. Since the segment $\gamma_1 = \gamma_2 = 0$, $p \in [0, p_0]$ is the edge of the stability boundary, the limit of the critical load, when damping tends to zero, $p_0^e = \lim_{\varepsilon\to 0} p_{cr}(\gamma_1, \gamma_2)$ for a fixed direction (e_1, e_2) is equal to the value of p at which the vector $e = (e_1, e_2, 0)$ leaves the tangent cone $K_{F_3}(p)$ with the increase of p from zero. In this case either the condition $(r_1(p_0^e), e) = 0$ or $(r_2(p_0^e), e) = 0$ is fulfilled. For example, at $\gamma_1 = \varepsilon$, $\gamma_2 = 0$ we have $e = (1, 0, 0)$, $p_0^e = 2$, $r_2(2) = (0, -5/2, 0)$, $(r_2(2), e) = 0$. From this it is seen that the limit of the critical load p_0^e is different for various directions (e_1, e_2). For all $(e_1, e_2) \neq c(4+5\sqrt{2}, 1)$, $c > 0$, this limit is less then p_0. At $(e_1, e_2) = c(4+5\sqrt{2}, 1)$, $c > 0$, we have $p_0^e = p_0$. It is connected with the fact that the directions $c(4 + 5\sqrt{2}, 1, \alpha)$, $\alpha \leq 0$, $c > 0$, belong to the tangent cone K_{G_2} from (6.8).

Degeneration of the dihedral angle at the deadlock of an edge singular point geometrically illustrates the effects of destabilization of a nonconservative system by small dissipative forces [8] and uncertainty of the critical load when damping parameters tend to zero [14]. Similar effects should be expected for other systems with deadlock of an edge and break of an edge singularities of the stability boundary.

7. Family of polynomials. Consider a linear homogeneous differential equation of the order m

$$x^{(m)} + a_1 x^{(m-1)} + \cdots + a_m x = 0 \tag{7.1}$$

whose coefficients $a_j \in \mathbf{R}$, $j = 1, 2, \ldots, m$ smoothly depend on a vector of parameters $p \in \mathbf{R}^n$. The characteristic equation for (7.1) is

$$\lambda^m + a_1 \lambda^{m-1} + \cdots + a_m = 0. \tag{7.2}$$

The trivial solution of (7.1) is asymptotically stable if and only if every root of (7.2) has a negative real part $\mathrm{Re}\lambda < 0$.

The stability domain boundary of a generic one-parameter family of polynomials ($n = 1$) is characterized by a simple root $\lambda = 0$ or a pair of simple imaginary roots $\lambda = \pm i\,\omega$. The stability boundary of a generic two-parameter (three-parameter) family consists of smooth curves (surfaces), corresponding to simple roots $\lambda = 0$ or $\lambda = \pm i\,\omega$, whose only singularities are characterized by the following strata [10]:

$$\begin{aligned} n = 2: &\quad \widehat{F}_1(0^2),\ \widehat{F}_2(0, \pm i\,\omega),\ \widehat{F}_3(\pm i\,\omega_1, \pm i\,\omega_2), \\ n = 3: &\quad \widehat{F}_1(0^2),\ \widehat{F}_2(0, \pm i\,\omega),\ \widehat{F}_3(\pm i\,\omega_1, \pm i\,\omega_2),\ \widehat{G}_1(0^3), \\ &\quad \widehat{G}_2\big((\pm i\,\omega)^2\big), \widehat{G}_3(0^2, \pm i\,\omega),\ \widehat{G}_4(0, \pm i\,\omega_1, \pm i\,\omega_2), \\ &\quad \widehat{G}_5(\pm i\,\omega_1, \pm i\,\omega_2, \pm i\,\omega_3), \end{aligned} \tag{7.3}$$

where all imaginary roots at the singular point are taken in brackets with a power denoting the multiplicity of a root. Other singularities disappear under an arbitrary small deformation of the family.

Introducing the vector $y \in \mathbf{R}^m$, with the components $y_i = x^{(i-1)}$, $i = 1, 2, \ldots, m$, (7.1) takes the form

$$\dot{y} = Ay, \tag{7.4}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \cdots & -a_1 \end{pmatrix}. \tag{7.5}$$

The characteristic equation for the matrix (7.5) is identically equal to (7.2); hence the stability domains for (7.1) and (7.4) coincide. For every root λ_0 of (7.2) there exists precisely one corresponding eigenvector u_0,

$$u_0 = c(1, \lambda_0, \lambda_0^2, \ldots, \lambda_0^{m-1}), \quad c = \text{const.}$$

For every multiple root λ_0 there exists one corresponding Jordan chain with the length equal to the multiplicity of λ_0. It is easy to see that in the case of generic one-, two-, and three-parameter families the singularities of the stability boundary $\widehat{F}_j$,

$\widehat{G}_s$, $j=1,2,3$, $s=1,2,\ldots,5$, for polynomials coincide with the singularities F_j, G_s, $j=1,2,3$, $s=1,2,\ldots,5$, for matrices, studied above. All the results for tangent cones and other geometric characteristics of singularities, obtained for families of matrices, are directly transferred to families of polynomials. The tangent cones to the stability domain of (7.1) at singular points (7.3) are determined in (2.4)–(2.6), (4.3)–(4.5), (4.10), (5.13), where the vectors r_j, f_j, q_j, h_j are calculated by the formulae (1.8), (1.17), (1.25) for the matrix (7.5).

Let us find the expressions for these vectors by means of the coefficients a_j, $j=1,2,\ldots,m$, of the equation (7.1). Denoting $Q(\lambda,p)=\lambda^m+a_1\lambda^{m-1}+\cdots+a_m$ and differentiating the equation $Q(\lambda,p)=0$ for a simple root λ ($Q=0$, $\partial Q/\partial\lambda\neq 0$) we have

$$\frac{\partial Q}{\partial\lambda}d\lambda+(\nabla Q,dp)=0,$$

$$\nabla\lambda=-\nabla Q\Big/\frac{\partial Q}{\partial\lambda},$$

where ∇ is the gradient operator

$$\nabla=\left(\frac{\partial}{\partial p_1},\frac{\partial}{\partial p_2},\ldots,\frac{\partial}{\partial p_n}\right)^T.$$

Recall that the vector r, determined in (1.8), is the gradient of the real part of a simple eigenvalue with respect to p. Hence,

$$r=-\mathrm{Re}\left(\nabla Q\Big/\frac{\partial Q}{\partial\lambda}\right). \tag{7.6}$$

Substituting the expression for $Q(\lambda,p)$ into (7.6) we obtain

$$r=-\mathrm{Re}\,\frac{\displaystyle\sum_{j=1}^{m}\nabla a_j\lambda^{m-j}}{\displaystyle m\lambda^{m-1}+\sum_{j=1}^{m-1}(m-j)a_j\lambda^{m-j-1}}. \tag{7.7}$$

For the simple root $\lambda=0$ ($a_m=0$, $a_{m-1}\neq 0$) it gives

$$r=-\frac{\nabla a_m}{a_{m-1}}. \tag{7.8}$$

Consider a double root λ_0. The vectors f_j, q_j, $j=1,2$, determined by (1.17), describe collapse of a double eigenvalue with the Jordan chain (1.10). Expanding $Q(\lambda,p)$ in Taylor series in the neighborhood of $\lambda=\lambda_0$, $p=p_0$ we have

$$\begin{aligned}Q(\lambda,p)&=(\nabla Q,\Delta p)+\frac{1}{2}\frac{\partial^2Q}{\partial\lambda^2}\Delta\lambda^2+\left(\nabla\frac{\partial Q}{\partial\lambda},\Delta p\right)\Delta\lambda\\&\quad+\frac{1}{2}\Delta p^T\nabla\nabla^TQ\Delta p+\frac{1}{6}\frac{\partial^3Q}{\partial\lambda^3}\Delta\lambda^3+\cdots,\end{aligned} \tag{7.9}$$

$$\Delta\lambda=\lambda-\lambda_0,\quad \Delta p=p-p_0.$$

Here the equalities $Q = 0$, $\partial Q/\partial\lambda = 0$ at $\lambda = \lambda_0$, $p = p_0$, determining a double root, were used. We substitute the perturbation of p in the form $\Delta p = \varepsilon e + O(\varepsilon^2)$ and the expansion for λ (1.13) into (7.9) and then equate the coefficients at ε and $\varepsilon^{3/2}$ zero. As the result, for the coefficient at ε we get

$$(\nabla Q, e) + \frac{1}{2}\frac{\partial^2 Q}{\partial\lambda^2}\lambda_1^2 = 0,$$

$$\lambda_1^2 = -\frac{2(\nabla Q, e)}{\dfrac{\partial^2 Q}{\partial\lambda^2}}.$$

If $\lambda_1^2 \neq 0$, then equating the coefficient at $\varepsilon^{3/2}$ zero we obtain

$$\lambda_2 = -\frac{\left(\nabla\dfrac{\partial Q}{\partial\lambda}, e\right) + \dfrac{1}{6}\dfrac{\partial^3 Q}{\partial\lambda^3}\lambda_1^2}{\dfrac{\partial^2 Q}{\partial\lambda^2}}$$

$$= \frac{\dfrac{1}{3}\dfrac{\partial^3 Q}{\partial\lambda^3}(\nabla Q, e) - \dfrac{\partial^2 Q}{\partial\lambda^2}\left(\nabla\dfrac{\partial Q}{\partial\lambda}, e\right)}{\left(\dfrac{\partial^2 Q}{\partial\lambda^2}\right)^2}.$$

Note that $\partial^2 Q/\partial\lambda^2 \neq 0$ for a double root. Thus, for the vectors f_j, q_j, $j = 1, 2$, we get the following expressions:

$$f_1 + i\,q_1 = -\frac{2\nabla Q}{\dfrac{\partial^2 Q}{\partial\lambda^2}},$$

$$f_2 + i\,q_2 = \frac{\dfrac{2}{3}\dfrac{\partial^3 Q}{\partial\lambda^3}\nabla Q - 2\dfrac{\partial^2 Q}{\partial\lambda^2}\nabla\dfrac{\partial Q}{\partial\lambda}}{\left(\dfrac{\partial^2 Q}{\partial\lambda^2}\right)^2}. \tag{7.10}$$

Substituting the explicit form of $Q(\lambda, p)$ into (7.10), by analogy with (7.7), the expression (7.10) can be written by means of the coefficients $a_j(p)$, $j = 1, 2, \ldots, m$, and λ_0. In the case of the double root $\lambda_0 = 0$ we get

$$f_1 = -\frac{\nabla a_m}{a_{m-2}},$$

$$f_2 = \frac{a_{m-3}\nabla a_m - a_{m-2}\nabla a_{m-1}}{a_{m-2}^2}. \tag{7.11}$$

The vectors h_j, $j = 1, 2, 3$, used for determining the tangent cone K_{G_1}, are needed only for $\lambda_0 = 0$. In this case $a_m = a_{m-1} = a_{m-2} = 0$, $a_{m-3} \neq 0$. The left and right eigenvectors and the associated vectors of the matrix (7.5), corresponding to the triple zero root and satisfying the normalization conditions (1.21), are

$$u_0 = (1, 0, 0, 0, \ldots, 0)^T,$$

$$u_1 = (0,1,0,0,\dots,0)^T,$$

$$u_2 = (0,0,1,0,\dots,0)^T,$$

$$v_0 = \left(0,0,1,*,\dots,*,\frac{1}{a_{m-3}}\right)^T,$$

$$v_1 = \left(0,1,0,*,\dots,*,-\frac{a_{m-4}}{a_{m-3}^2}\right)^T,$$

$$v_2 = \left(1,0,0,*,\dots,*,\frac{a_{m-4}^2 - a_{m-3}a_{m-5}}{a_{m-3}^3}\right)^T,$$

where asterisks denote the components, which don't affect the resultant expressions. Substitution of these vectors into (1.25) gives

$$h_1 = -\frac{\nabla a_m}{a_{m-3}},$$

$$h_2 = \frac{a_{m-4}\nabla a_m - a_{m-3}\nabla a_{m-1}}{a_{m-3}^2}, \tag{7.12}$$

$$h_3 = \frac{(a_{m-3}a_{m-5} - a_{m-4}^2)\nabla a_m + a_{m-3}a_{m-4}\nabla a_{m-1} - a_{m-3}^2\nabla a_{m-2}}{a_{m-3}^3}.$$

Note that in the case of the generic family of polynomials, vectors determining tangent cones to the stability domain at a singular point are linearly independent.

Example. As an example let us consider the stability problem from section 3. The characteristic equation has the form (3.3). The system is considered as dependent on two parameters R and ρ. At the point $R_0 = \sqrt{L/C}$, $\rho_0 = -\sqrt{L/C}$ the characteristic equation (3.3) has the double zero root corresponding to the singularity $\widehat{F}_1(0^2)$. The vectors f_1 and f_2, calculated with the use of (7.11), are

$$f_1 = \frac{1}{L\sqrt{LC}}\begin{pmatrix}-1\\-1\end{pmatrix}, \quad f_2 = \frac{1}{L}\begin{pmatrix}1\\-1\end{pmatrix}.$$

These expressions coincide with (3.4), where these vectors were calculated using (1.17). The tangent cone to the stability domain $K_{\widehat{F}_1}$ at the point under consideration has the form (2.4); see Figure 3.1.

Also, the results (6.6)–(6.8) on singularities of the stability boundary for Ziegler's pendulum can be derived using polynomial formulation (6.4).

8. Concluding remarks. Two methods of investigation of singularities are developed in this paper. They are constructive and convenient for numerical implementation. These methods can be applied for studying other types of singularities (also nongeneric) of the stability domain of systems depending on an arbitrary number of parameters. The first method connected with expansions of eigenvalues can be applied if the tangent cone of the singularity does not belong to the set of directions violating the condition Γ [16] (for nonderogatory eigenvalue of the matrix A this condition is ${v_0}^T A_1 u_0 \neq 0$):

$$K \not\subset \{e : \Gamma \text{ is not satisfied}\}.$$

In this set of directions the expansions of eigenvalues in powers of $\varepsilon^{1/l}$ (l is a multiplicity) are not valid. For example, this method can be applied to the singularities, where there are only simple and double eigenvalues with zero real part. Singularities investigated in sections 2 and 4 are just of this type.

The second method connected with versal deformations can be useful for investigation of singularities determined by pure imaginary eigenvalues with higher multiplicities. The main point here is to find appropriate vectors, like h_1, h_2, h_3 in section 5, which connect parameter space of the problem with parameter space of the versal deformation.

As an example of application of the second method let us consider the singular point of the stability domain determined by zero eigenvalue with one Jordan block of the order k. Let $u_0, u_1, \ldots, u_{k-1}$ and $v_0, v_1, \ldots, v_{k-1}$ be corresponding Jordan chains of right and left eigenvectors and associated vectors satisfying normalization conditions $v_0^T u_{k-1} = 1$, $v_i^T u_{k-1} = 0$, $i = 1, \ldots, k-1$. Then introducing the vectors h_i, $i = 1, 2, \ldots, k$, with components defined by the formulae

$$h_i^j = \sum_{s=0}^{i-1} v_s^T \frac{\partial A}{\partial p_j} u_{i-s-1}, \quad i = 1, 2, ..., k, \;\; j = 1, 2, ..., n,$$

and making the same steps as in the section 5, in the case of linearly independent vectors h_i, $i = 1, 2, \ldots, k$, we get the expression for the tangent cone K_0 to the stability domain at this singular point:

$$K_0 = \{\, e : (h_1, e) = \cdots = (h_{k-2}, e) = 0, \; (h_{k-1}, e) \leq 0, \; (h_k, e) \leq 0 \,\}.$$

Note that for $k = 3$ these expressions are the same as the expressions which have been found in section 5 for the singularity G_1. Evidently, to fulfill the linear independence condition we need $n \geq k$, i.e., the dimension of the parameter space must be greater than or equal to the multiplicity of zero eigenvalue. If $n \geq k$, then the vectors h_i, $i = 1, 2, \ldots, k$, are linearly independent for the generic family of matrices.

To investigate singularities of a boundary of the stability domain of a family of polynomials, first we have to consider the singularity of the corresponding family of matrices (7.5) and then express the result in terms of the coefficients of the polynomial and their derivatives with respect to parameters.

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