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Short Communication

Conditions revisited for asymptotic stability of pervasive damped linear systems

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Abstract

A linear oscillatory system with a positive definite mass matrix **M**, a positive definite stiffness matrix **K** and a positive semi-definite damping matrix **D** can be asymptotically stable. In this case the damping **D** of the system is called pervasive. The condition for the damping to be pervasive was given in a paper by Moran in 1970. Moran's result has been reinvented in a recent paper from 2004 motivated by a long discussion on "intuitive" examples of marginal and asymptotic stability of systems with incomplete damping. The aim of the present note is to draw attention of the readers to Moran's result. We reveal the importance of Moran's criterion for suppression of oscillations by incomplete damping in systems with multiple eigenfrequencies. Finally, we provide extensions to the case of indefinite damping.

In 1970 Moran [1] published a paper investigating whether a system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0, \quad \mathbf{x} \in \mathbb{R}^n \tag{1}$$

with positive definite matrices **M** and **K** ($\mathbf{M} > 0, \mathbf{K} > 0$) and a positive semi-definite damping matrix **D** ($\mathbf{D} \geq 0$) is *asymptotically stable* (then the damping is often called pervasive) or has at least one harmonic solution (this case is often called marginally stable or having a residual motion). The system is asymptotically stable if all the eigenvalues λ of the problem $(\lambda^2\mathbf{M} + \lambda\mathbf{D} + \mathbf{K})\mathbf{u} = 0$ have negative real parts. Marginal stability corresponds to the existence of purely imaginary eigenvalues λ.

Moran proved the following: A necessary and sufficient condition for system (1) to be asymptotically stable is that none of the eigenvectors (eigenmodes) $\mathbf{v}$ of the conservative system $(\lambda^2\mathbf{M} + \mathbf{K})\mathbf{v} = 0$ lies in the null space of **D**, that is $\mathbf{D}\mathbf{v} \neq 0$ for all eigenvectors $\mathbf{v}$. The Moran criterion provides a clear physical explanation: the system is asymptotically stable if and only if damping terms couple all the eigenmodes of the conservative system. Pervasive damping was also recognized by Müller [2] as essential for an extension of the Thomson–Tait–Chetayev stability criterion. Of course, other criteria for asymptotic stability of (1) exist. For

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example, control theory provides the rank condition [2]

$$\mathrm{rank}[\mathbf{M}^{-1}\mathbf{D}, (\mathbf{M}^{-1}\mathbf{K})(\mathbf{M}^{-1}\mathbf{D}), \ldots, (\mathbf{M}^{-1}\mathbf{K})^{n-1}(\mathbf{M}^{-1}\mathbf{D})] = n. \tag{2}$$

Recently Shahruz and Kessler [3] published a paper whose central result is the above Moran criterion, except for two superfluous assumptions the authors introduced. They were not aware of the original work and used a perturbation technique instead of the linear algebra approach of Moran. The paper of Shahruz and Kessler [3] was motivated by a long discussion of several "intuitive" examples of marginal and asymptotic stability in systems with incomplete damping [4–6], together with the answers [7–9] based on the rank criterion (2). Moran's result was not mentioned is those papers as well.

The aim of our letter is to draw attention of the readers to Moran's result, which seems to be forgotten (even though it was mentioned in classical textbooks [2,10]). We argue that the "intuition" in the examples discussed in Refs. [4–6] gets an immediate rigorous explanation by applying Moran's criterion: in each case, one can easily see the eigenmode which is not damped (in marginally stable systems) or all the eigenmodes are coupled by damping (in asymptotically stable systems). Below we reveal the importance of Moran's result for describing stabilization by incomplete damping in systems with multiple eigenfrequencies. Then we provide extensions to the case of indefinite damping.

The problem of asymptotic stability in systems with incomplete damping is important in many practical problems. This corresponds to systems with negligibly small natural damping forces, whose oscillations must be actively damped, e.g., with dashpots. Clearly, one dashpot provides dissipative forces in one dimension (one degree of freedom). Therefore, one is naturally interested in effective damping with smaller rank damping matrix $\mathbf{D}$ (fewer dashpots). The following statement provides the lower limit for the rank.

Theorem 1. *Let m be the maximal multiplicity of eigenfrequencies of the conservative system* (*with* $\mathbf{D} = 0$). *Then,*

(a) *the system with incomplete damping is marginally stable* (*not asymptotically stable*) *if* $\mathrm{rank}\,\mathbf{D} < m$;
(b) *the system with incomplete damping is asymptotically stable for almost all damping matrices such that* $\mathrm{rank}\,\mathbf{D} \geq m$ (*i.e., within a set of matrices* $\mathbf{D}$ *satisfying the condition* $\mathrm{rank}\,\mathbf{D} = m'$ *for any fixed* $m' \geq m$, *the damping matrices corresponding to marginally stable systems form a zero-measure subset*).

The theorem follows directly from the Moran criterion.

Multiple eigenfrequencies are typical in oscillatory systems with symmetry or as a result of optimization. This makes multiple eigenfrequencies important in the analysis of oscillations as well as wave dynamics. The maximal multiplicity of eigenfrequencies m gives the number of active dampers necessary for suppression of oscillations. Notice that the maximal multiplicity of eigenvalues in a symmetric system may be unbounded: for example, the eigenfrequencies of an elastic sphere with fixed boundary have multiplicities $2n+1$ for any integer n, see e.g. Ref. [11].

In the following we will contribute with an extension. Consider system (1), but this time with an *indefinite* damping matrix $\mathbf{D}$. Indefinite damping matrices can appear in modelling sliding bearings, in investigating cutting of metals where self-excited vibrations are the result of dry friction, and in modelling squealing of car brakes, see e.g. Refs. [12–14]. In the case of an indefinite damping matrix, system (1) may be stable or unstable. Well-known *necessary* (but not sufficient) conditions for stability are $\mathrm{tr}(\mathbf{M}^{-1}\mathbf{D}) \geqslant 0$ and $\mathrm{tr}(\mathbf{K}^{-1}\mathbf{D}) \geqslant 0$, whereas the signs $\geqslant$ have to be sharpened for asymptotic stability. In the case of sufficiently small damping a first-order perturbation approach shows that a necessary and sufficient condition for asymptotic stability is $\mathbf{v}^*\mathbf{D}\mathbf{v} > 0$ for all the eigenvectors $\mathbf{v}$ of the conservative system $(\lambda^2\mathbf{M} + \mathbf{K})\mathbf{v} = 0$ [16]. But system (1) with an indefinite damping matrix $\mathbf{D}$ will always become unstable, if $\mathbf{D}$ is multiplied by a sufficient large factor, see Ref. [15].

Consider now a stable system (1) with an indefinite damping matrix. Assume that an eigenvector $\mathbf{v}$ of the problem $(\lambda^2\mathbf{M} + \mathbf{K})\mathbf{v} = 0$ satisfies $\mathbf{D}\mathbf{v} = 0$. Then $\mathbf{v}$ is also an eigenvector of $(\lambda^2\mathbf{M} + \lambda\mathbf{D} + \mathbf{K})\mathbf{v} = 0$ with the same purely imaginary eigenvalue λ, which means the existence of a residual motion. So asymptotic stability implies $\mathbf{D}\mathbf{v} \neq 0$ for all eigenvectors $\mathbf{v}$ of the conservative system, i.e., all the eigenmodes must be coupled via damping terms. Thus, one part of the Moran criterion is extended to the case of indefinite damping. The opposite direction of the Moran criterion does not work for indefinite damping.

Example 1. Consider an oscillatory system (1) with

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \varepsilon \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 3 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \tag{3}$$

where the damping matrix is indefinite. The eigenmodes of the conservative system $(\lambda^2\mathbf{M} + \mathbf{K})\mathbf{v} = 0$ are

$$\mathbf{v}_1 = [-1, -2, 2]^{\mathrm{T}}, \quad \mathbf{v}_2 = [-2, 2, 1]^{\mathrm{T}}, \quad \mathbf{v}_3 = [2, 1, 2]^{\mathrm{T}}. \tag{4}$$

Then $\mathbf{v}_1^*\mathbf{D}\mathbf{v}_1 = 11\varepsilon > 0$, $\mathbf{v}_2^*\mathbf{D}\mathbf{v}_2 = 5\varepsilon > 0$ and $\mathbf{v}_3^*\mathbf{D}\mathbf{v}_3 = 2\varepsilon > 0$, and the system is asymptotically stable for sufficiently small ε. Numerical analysis shows that the system is asymptotically stable for $0 < \varepsilon < 0.626$, and unstable for $\varepsilon > 0.626$.

As a second extension, consider a gyroscopic system

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0 \tag{5}$$

with $\mathbf{M} > 0$, $\mathbf{K} > 0$, $\mathbf{D} \geqslant 0$ and a skew-symmetric matrix of gyroscopic forces $\mathbf{G}$ ($\mathbf{G} = -\mathbf{G}^{\mathrm{T}}$). According to Moran [1], system (5) is asymptotically stable if and only if none of the eigenvectors $\mathbf{v}$ of the conservative (undamped) system $(\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K})\mathbf{v} = 0$ lie in the nullspace of $\mathbf{D}$. Again, we can extend this theory to the case of an *indefinite* damping matrix $\mathbf{D}$. Let us assume that the system (5) with indefinite matrix is stable (this system may be stable even if the damped system without gyroscopic forces is unstable, see Refs. [16,17]). Now we can ask whether this stable system is asymptotically stable. Again only one part of the Moran criterion holds. If an eigenvector $\mathbf{v}$ of the conservative system $(\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K})\mathbf{v} = 0$ satisfies $\mathbf{D}\mathbf{v} = 0$, then $\mathbf{v}$ is also an eigenvector of $(\lambda^2\mathbf{M} + \lambda(\mathbf{D} + \mathbf{G}) + \mathbf{K})\mathbf{v} = 0$ with the same purely imaginary eigenvalue λ. Therefore, asymptotic stability implies $\mathbf{D}\mathbf{v} \neq 0$ for all eigenvectors $\mathbf{v}$ of conservative gyroscopic system. The reverse is not true for indefinite damping.

We can actually say a little more if the indefinite damping is sufficiently small. A perturbation approach results in the following: The system $\mathbf{M}\ddot{\mathbf{x}} + (\varepsilon\mathbf{D} + \mathbf{G})\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0$ is asymptotically stable for sufficiently small $\varepsilon > 0$ if and only if $\mathbf{v}^*\mathbf{D}\mathbf{v} > 0$ for all the eigenvectors $\mathbf{v}$ of the conservative system $(\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K})\mathbf{v} = 0$. This is demonstrated in the following example.

Example 2. Consider an oscillatory system (5) with the matrices (3) and

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}. \tag{6}$$

The eigenmodes of the conservative system $(\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K})\mathbf{v} = 0$ are

$$\begin{aligned} \mathbf{v}_1 &= [-0.1914\mathrm{i}, -0.0256 + 0.0641\mathrm{i}, 0.1819 - 0.0197\mathrm{i}]^{\mathrm{T}}, \\ \mathbf{v}_2 &= [-0.2687 + 0.2271\mathrm{i}, -0.4828 - 0.2091\mathrm{i}, 0.5443]^{\mathrm{T}}, \\ \mathbf{v}_3 &= [0.1478 - 0.0867\mathrm{i}, -0.0000 - 0.3322\mathrm{i}, -0.0553 - 0.1352\mathrm{i}]^{\mathrm{T}}, \end{aligned} \tag{7}$$

corresponding to the eigenvalues $\lambda_1 = 3.5152\mathrm{i}$, $\lambda_2 = 0.6596\mathrm{i}$, $\lambda_3 = 2.2822\mathrm{i}$ (complex conjugate eigenvectors $\bar{\mathbf{v}}_j$ correspond to complex conjugate eigenvalues $\bar{\lambda}_j$, $j = 1, 2, 3$). Then $\mathbf{v}_1^*\mathbf{D}\mathbf{v}_1 = 0.0064\varepsilon > 0$, $\mathbf{v}_2^*\mathbf{D}\mathbf{v}_2 = 0.7262\varepsilon > 0$ and $\mathbf{v}_3^*\mathbf{D}\mathbf{v}_3 = 0.2127\varepsilon > 0$, and the system is asymptotically stable for sufficiently small ε. Numerical analysis shows that the system is asymptotically stable for $0 < \varepsilon < 0.8910$, and unstable for $\varepsilon > 0.8910$.

References

[1] T.J. Moran, A simple alternative to the Routh–Hurwitz criterion for symmetric systems, *ASME Journal of Applied Mechanics* 37 (1970) 1168–1170.

[2] P.C. Müller, *Stabilität und Matrizen*, Springer, Berlin, 1977.

[3] S.M. Shahruz, P. Kessler, Residual motion in damped linear systems, *Journal of Sound and Vibration* 276 (2004) 1093–1100.

[4] E.V. Wilms, R.B. Pinkney, The finite residual motion of a damped two-degree-of-freedom vibrating system, *Journal of Sound and Vibration* 148 (1991) 533–534.
[5] E.V. Wilms, R.B. Pinkney, The finite residual motion of a damped three-degree-of-freedom vibrating system, *Journal of Sound and Vibration* 224 (1999) 937–940.
[6] E.V. Wilms, H. Cohen, The finite residual motion of a damped four-degree-of-freedom vibrating system, *Journal of Sound and Vibration* 244 (2001) 173–176.
[7] M. Gürgöze, Another solution to "The finite residual motion of a damped two-degree-of-freedom vibrating system", *Journal of Sound and Vibration* 224 (1994) 409–410.
[8] M. Gürgöze, Alternative solution to "The finite residual motion of a damped four-degree-of-freedom vibrating system", *Journal of Sound and Vibration* 255 (2002) 987–988.
[9] P. Hagedorn, A note on "The finite residual motion of a damped two-degree-of-freedom vibrating system", *Journal of Sound and Vibration* 232 (2000) 834–835.
[10] K. Huseyin, *Vibrations and Stability of Multiple Parameter Systems*, Noordhoff, Alphen aan den Rijn, 1978.
[11] A.N. Tikhonov, A.A. Samarskii, *Equations of Mathematical Physics*, Dover Publishers, New York, 1990.
[12] P. Hagedorn, Neuere Entwicklungen in der Technischen Schwingungslehre, *Zeitschrift fur Angewandte Mathematik und Mechanik* 68 (1988) T22–T31.
[13] R.N. Arnold, L. Maunder, *Gyrodynamics and its Engineering Applications*, Academic Press, New York, 1961.
[14] K. Popp, M. Rudolph, M. Kröger, M. Lindner, *Mechanisms to Generate and to Avoid Friction Induced Vibrations*, VDI-Berichte 1736, VDI-Verlag Dusseldorf 2002, pp. 1–15.
[15] P. Freitas, M. Grinfeld, P.A. Knight, Stability of finite-dimensional systems with indefinite damping, *Advances in Mathematical Sciences and Application* 7 (1997) 435–446.
[16] W. Kliem, P.C. Müller, Gyroscopic stabilization of indefinite damped systems, *Zeitschrift fur Angewandte Mathematik und Mechanik* 77 (1997) 163–164.
[17] W. Kliem, C. Pommer, Lyapunov functions and solutions of the Lyapunov matrix equation for marginally stable systems, in: I. Gohberg, H. Langer (Eds.), *Operator Theory*: *Advances and Applications*, Vol. 130, Birkhäuser, Basel, 2001.