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On the Weierstrass Preparation Theorem

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Abstract—An analytic function of several variables is considered. It is assumed that the function vanishes at some point. According to the Weierstrass preparation theorem, in the neighborhood of this point the function can be represented as a product of a nonvanishing analytic function and a polynomial in one of the variables. The coefficients of the polynomial are analytic functions of the remaining variables. In this paper we construct a method for finding the nonvanishing function and the coefficients of the polynomial in the form of Taylor series whose coefficients are found from an explicit recursive procedure using the derivatives of the initial function. As an application, an explicit formula describing a bifurcation diagram locally up to second-order terms is derived for the case of a double root.

Key words: *Weierstrass preparation theorem, analytic function of several variables, bifurcation diagram.*

Consider an analytic function of several variables $f(z,\boldsymbol{p})$, $z \in \mathbb{C}$, $\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{C}^n$, which vanishes at the point z_0, $\boldsymbol{p}_0 = (p_1^0, \dots, p_n^0)$. By the Weierstrass preparation theorem [1, 2], in the neighborhood of the point $z_0, \boldsymbol{p}_0$ the function $f(z,\boldsymbol{p})$ can be expressed uniquely as a product of two analytic functions, one of which is a polynomial in $z - z_0$ with leading coefficient equal to one, and the other function is not equal to zero at the point $z = z_0$, $\boldsymbol{p} = \boldsymbol{p}_0$. The representation $f(z,\boldsymbol{p})$ as a product of two factors simplifies the problem of studying the roots of the equation $f(z,\boldsymbol{p}) = 0$ and their properties. For example, if $f(z,\boldsymbol{p})$ is a polynomial in z, then it follows from the Weierstrass preparation theorem that we can essentially lower the order of the equation in studying the dependence of isolated roots of the polynomial $f(z,\boldsymbol{p})$ on $\boldsymbol{p}$. Such an operation is especially useful in the analysis of multiple roots in problems of stability, in which one tries to define the domain stability (the set of values of the parameters $\boldsymbol{p}$ for which the roots $z(\boldsymbol{p})$ satisfy a certain condition) [3]. Another area of research where this theorem can be effectively used is the study of bifurcation diagrams (the sets of values of the parameters $\boldsymbol{p}$ for which there are multiple roots).

The applications described above can be realized under the condition that the factors into which the function $f(z,\boldsymbol{p})$ is expanded according to the preparation theorem are known. In the present paper, we construct a method for finding these factors in the form of Taylor series whose coefficients are found from an explicit recursive procedure using the derivatives of the function $f(z,\boldsymbol{p})$ at the point $z = z_0$, $\boldsymbol{p} = \boldsymbol{p}_0$. As an application, an explicit formula describing a bifurcation diagram locally up to second-order terms is derived for the case of a double root.

The preparation theorem for the case of infinitely differentiable functions was proved by Malgrange [4]. The results of the present paper can be carried over without changes to this case and enable us to find the Taylor series of the desired functions.

Weierstrass Preparation Theorem [1, 2]**.** *Suppose that $f(z,\boldsymbol{p})$ is an analytic function vanishing at the point $z = z_0$, $\boldsymbol{p} = \boldsymbol{p}_0$, where $z = z_0$ is an m-multiple root of the equation $f(z,\boldsymbol{p}_0) = 0$, i.e.,*

$$f(z_0,\boldsymbol{p}_0) = \frac{\partial f}{\partial z} = \dots = \frac{\partial^{m-1} f}{\partial z^{m-1}} = 0, \qquad \frac{\partial^m f}{\partial z^m} \neq 0,$$

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where the derivatives are taken at the point $z = z_0$, $\boldsymbol{p} = \boldsymbol{p}_0$. *Then there exists a neighborhood* $U_0 \subset \mathbb{C}^{n+1}$ *of the point* $(z_0, \boldsymbol{p}_0)$ *in which the function* $f(z, \boldsymbol{p})$ *can be expressed as*

$$f(z, \boldsymbol{p}) = \big((z - z_0)^m + a_{m-1}(\boldsymbol{p})(z - z_0)^{m-1} + \dots + a_0(\boldsymbol{p})\big)b(z, \boldsymbol{p}), \tag{1}$$

where $a_0(\boldsymbol{p}), \dots, a_{m-1}(\boldsymbol{p}), b(z, \boldsymbol{p})$ *are analytic functions uniquely defined by the function* $f(z, \boldsymbol{p})$ *and* $a_i(\boldsymbol{p}_0) = 0$, $b(z_0, \boldsymbol{p}_0) \neq 0$.

To obtain the expansion (1), it is necessary to determine the functions $a_i(\boldsymbol{p})$ and $b(z, \boldsymbol{p})$. In the neighborhood of the point $z = z_0$, $\boldsymbol{p} = \boldsymbol{p}_0$, the functions $a_i(\boldsymbol{p})$ and $b(z, \boldsymbol{p})$ can be expressed as the Taylor series

$$a_i(\boldsymbol{p}) = \sum_{\boldsymbol{h}} \frac{1}{\boldsymbol{h}!} a_{i,\boldsymbol{h}} \Delta \boldsymbol{p}^{\boldsymbol{h}}, \qquad b(z, \boldsymbol{p}) = \sum_{k,\boldsymbol{h}} \frac{1}{k!\,\boldsymbol{h}!} b_{k,\boldsymbol{h}} \Delta z^k \Delta \boldsymbol{p}^{\boldsymbol{h}}, \tag{2}$$

where the sums are taken over all $k \in \mathbb{Z}_+$, $\boldsymbol{h} = (h_1, \dots, h_n)$, $h_i \in \mathbb{Z}_+$ ($\mathbb{Z}_+$ is the set of nonnegative integers) and the following notation is used:

$$a_{i,\boldsymbol{h}} = \frac{\partial^{|\boldsymbol{h}|} a_i}{\partial p_1^{h_1} \cdots \partial p_n^{h_n}}, \qquad b_{k,\boldsymbol{h}} = \frac{\partial^{k+|\boldsymbol{h}|} b}{\partial z^k \partial p_1^{h_1} \cdots \partial p_n^{h_n}}, \qquad |\boldsymbol{h}| = h_1 + \dots + h_n,$$

$$\Delta \boldsymbol{p}^{\boldsymbol{h}} = \prod_{j=1}^{n} (p_j - p_j^0)^{h_j}, \qquad \Delta z^k = (z - z_0)^k, \qquad \boldsymbol{h}! = h_1! \cdots h_n!.$$

All the derivatives are calculated for $z = z_0$, $\boldsymbol{p} = \boldsymbol{p}_0$. By the zeroth-order derivative we mean the value of the function at a point, in particular, $b_{k,0} = \partial^k b / \partial z^k$. The derivatives of the function $f(z, \boldsymbol{p})$ are denoted in a similar way. The definition of the functions $a_i(\boldsymbol{p})$, $b(z, \boldsymbol{p})$ is equivalent to the determination of the derivatives $a_{i,\boldsymbol{h}}$, $b_{k,\boldsymbol{h}}$ defining the coefficients of the expansion (2). In the following theorem, an explicit expression for the desired derivatives is given in terms of the derivatives of the function $f(z, \boldsymbol{p})$.

Theorem. *The derivatives* $a_{i,\boldsymbol{h}}$ *and* $b_{k,\boldsymbol{h}}$ *of the functions* $a_i(\boldsymbol{p}), b(z, \boldsymbol{p})$ *in relation* (1) *satisfy the following recurrence relations*:

$$a_{i,\boldsymbol{h}} = \sum_{j=0}^{i} \alpha_{ij} F_{j,\boldsymbol{h}}, \qquad F_{j,\boldsymbol{h}} = f_{j,\boldsymbol{h}} - \sum_{k=0}^{j} \sum_{\substack{\boldsymbol{h}'+\boldsymbol{h}''=\boldsymbol{h} \\ \boldsymbol{h}'\neq 0,\, \boldsymbol{h}''\neq 0}} c(j, k;\, \boldsymbol{h}', \boldsymbol{h}'') a_{k,\boldsymbol{h}'} b_{j-k,\boldsymbol{h}''}, \tag{3}$$

$$b_{k,\boldsymbol{h}} = \frac{k!}{(m+k)!} \Bigg[f_{m+k,\boldsymbol{h}} - \sum_{j=0}^{m-1} \sum_{\substack{\boldsymbol{h}'+\boldsymbol{h}''=\boldsymbol{h} \\ \boldsymbol{h}'\neq 0}} c(m+k, j;\, \boldsymbol{h}', \boldsymbol{h}'') a_{j,\boldsymbol{h}'} b_{m+k-j,\boldsymbol{h}''} \Bigg], \tag{4}$$

where the coefficients α_{ij} $(i \geq j)$, $c(k, j;\, \boldsymbol{h}', \boldsymbol{h}'')$ *are determined by the relations*

$$\alpha_{jj} = \frac{m!}{j!\, f_{m,0}}, \qquad \alpha_{ij} = -\frac{m!}{f_{m,0}} \sum_{k=j}^{i-1} \frac{f_{m+i-k,0}\, \alpha_{kj}}{(m+i-k)!} \quad (i > j),$$
$$c(j, k;\, \boldsymbol{h}', \boldsymbol{h}'') = \frac{j!}{(j-k)!} \prod_{s=1}^{n} \frac{(h_s' + h_s'')!}{h_s'!\, h_s''!}. \tag{5}$$

Proof. Relations (4) for $\boldsymbol{h}=0$ are of the form

$$b_{k,0}=\frac{k!\,f_{m+k,0}}{(m+k)!}. \tag{6}$$

They are obtained by $(m+k)$-fold differentiation of the identity $f(z,\boldsymbol{p}_0)=(z-z_0)^m b(z,\boldsymbol{p}_0)$ with respect to z. Let us take the derivative $\partial^{i+|\boldsymbol{h}|}/\partial z^i\partial p_1^{h_1}\cdots\partial p_n^{h_n}$ of both sides of relation (1):

$$f_{i,\boldsymbol{h}}=\sum_{j=0}^{\min(i,m)}\sum_{\boldsymbol{h}'+\boldsymbol{h}''=\boldsymbol{h}}c(i,j\,;\boldsymbol{h}',\boldsymbol{h}'')a_{j,\boldsymbol{h}'}b_{i-j,\boldsymbol{h}''},\qquad a_{m,\boldsymbol{h}'}=\begin{cases}1, & \boldsymbol{h}'=0,\\ 0, & \boldsymbol{h}'\neq 0.\end{cases} \tag{7}$$

Suppose that $i<m$. Expressing the summands containing $a_{j,\boldsymbol{h}}$ in terms of the other summands and using (6), we obtain

$$\sum_{j=0}^{i}\beta_{ij}a_{j,\boldsymbol{h}}=F_{i,\boldsymbol{h}},\qquad \beta_{ij}=\frac{i!\,f_{m+i-j,0}}{(m+i-j)!}, \tag{8}$$

where the $F_{i,\boldsymbol{h}}$ are defined in (3). Equations (8) for $i=0,\dots,m-1$ can be written in matrix form:

$$\mathbf{G}\begin{pmatrix}a_{0,\boldsymbol{h}}\\ \vdots\\ a_{m-1,\boldsymbol{h}}\end{pmatrix}=\begin{pmatrix}F_{0,\boldsymbol{h}}\\ \vdots\\ F_{m-1,\boldsymbol{h}}\end{pmatrix},\qquad \mathbf{G}=\begin{pmatrix}\beta_{00} & 0 & 0 & 0\\ \beta_{10} & \beta_{11} & 0 & 0\\ \vdots & \vdots & \ddots & 0\\ \beta_{(m-1)0} & \beta_{(m-1)1} & \cdots & \beta_{(m-1)(m-1)}\end{pmatrix}.$$

Since, by assumption, $\beta_{ii}=i!\,f_{m,0}/m!\neq 0$, we have $\det\mathbf{G}\neq 0$ and, therefore, the system of equations has a unique solution of the form

$$\begin{pmatrix}a_{0,\boldsymbol{h}}\\ \vdots\\ a_{m-1,\boldsymbol{h}}\end{pmatrix}=\mathbf{G}^{-1}\begin{pmatrix}F_{\boldsymbol{h},0}\\ \vdots\\ F_{\boldsymbol{h},m-1}\end{pmatrix},\qquad \mathbf{G}^{-1}=\begin{pmatrix}\alpha_{00} & 0 & 0\\ \vdots & \ddots & 0\\ \alpha_{(m-1)0} & \cdots & \alpha_{(m-1)(m-1)}\end{pmatrix}. \tag{9}$$

We can prove that the matrix $\mathbf{G}^{-1}$ is of the form (9) with entries α_{ij} defined in (5) by directly verifying the identity $\mathbf{G}\mathbf{G}^{-1}=\mathbf{I}$ ($\mathbf{I}$ is the unit matrix). Relation (9) is the expression for relations (3) of the theorem in matrix form.

Consider the case $i\geq m$. Then relation (4) of the theorem is obtained by rearranging the terms in relation (7) with regard to (6) and by substituting $m+k$ for i. □

Relations (3)–(5) are recurrence relations allowing us to find the derivatives of $a_i(\boldsymbol{p})$ and $b(z,\boldsymbol{p})$ from the derivatives of the function $f(z,\boldsymbol{p})$ in the following order:

$$b_{k,0}\to a_{i,\boldsymbol{h}},\qquad b_{k,\boldsymbol{h}}\ (|\boldsymbol{h}|=1)\ \to a_{i,\boldsymbol{h}},\qquad b_{k,\boldsymbol{h}}\ (|\boldsymbol{h}|=2)\ \to\cdots.$$

Here, to determine the derivative $a_{i,\boldsymbol{h}}$, it is necessary to calculate $a_{j,\boldsymbol{h}'},b_{j,\boldsymbol{h}'}$ for $j\leq i+(|\boldsymbol{h}-\boldsymbol{h}'|-1)m$ and $\boldsymbol{h}'<\boldsymbol{h}$ at the previous stages ($h_s'\leq h_s$ for all $s=1,\dots,n$ and $h_s'<h_s$ for some value of s). Thus the value of $a_{i,\boldsymbol{h}}$ can be determined from the derivatives $f_{k,\boldsymbol{h}''}$, where $k\leq i+|\boldsymbol{h}-\boldsymbol{h}''|m$ and $\boldsymbol{h}''\leq\boldsymbol{h}$.

The first steps of the recursive procedure are

$$a_i(\boldsymbol{p}_0)=0,\qquad \frac{\partial^k b}{\partial z^k}=\frac{k!}{(m+k)!}\frac{\partial^{m+k}f}{\partial z^{m+k}},\qquad \frac{\partial a_i}{\partial p_s}=\sum_{j=0}^{i}\alpha_{ij}\frac{\partial^{j+1}f}{\partial z^j\partial p_s},$$

$$\frac{\partial^{k+1}b}{\partial z^k\partial p_s}=\frac{k!}{(m+k)!}\left[\frac{\partial^{m+k+1}f}{\partial z^{m+k}\partial p_s}-\sum_{j=0}^{m-1}\frac{(m+k)!}{(m+k-j)!}\frac{\partial a_j}{\partial p_s}\frac{\partial^{m+k-j}b}{\partial z^{m+k-j}}\right], \tag{10}$$

$$\frac{\partial^2 a_i}{\partial p_s\partial p_t}=\sum_{j=0}^{i}\alpha_{ij}\left[\frac{\partial^{j+2}f}{\partial z^j\partial p_s\partial p_t}-\sum_{k=0}^{j}\frac{j!}{(j-k)!}\left(\frac{\partial a_k}{\partial p_s}\frac{\partial^{j-k+1}b}{\partial z^{j-k}\partial p_t}+\frac{\partial a_k}{\partial p_t}\frac{\partial^{j-k+1}b}{\partial z^{j-k}\partial p_s}\right)\right].$$

Example. The function $f(z, \boldsymbol{p})$ can be regarded as a family of functions of a single variable z, where $\boldsymbol{p}$ is the parameter vector.

By a *bifurcation diagram* we mean the set of values of the vector $\boldsymbol{p}$ for which the function $f(z, \boldsymbol{p})$ has multiple roots z. Consider the double root $z = z_0$. It follows from the Weierstrass preparation theorem that the surface in the space of $\boldsymbol{p}$ on which this root remains a double root (the value of the root may vary) can be determined by equating to zero the discriminant $D = (a_1(\boldsymbol{p}))^2 - 4a_0(\boldsymbol{p})$ of the equation $(z - z_0)^2 + a_1(\boldsymbol{p})(z - z_0) + a_0(\boldsymbol{p}) = 0$. Substituting the expansions (2) into the equation $D = 0$, we can write the equation of this surface as

$$D = -4\sum_{r=1}^{n} \frac{\partial a_0}{\partial p_r}(p_r - p_r^0) + \sum_{s,t=1}^{n} \left(\frac{\partial a_1}{\partial p_s}\frac{\partial a_1}{\partial p_t} - 2\frac{\partial^2 a_0}{\partial p_s \partial p_t} \right)(p_s - p_s^0)(p_t - p_t^0) + o\big(\|\boldsymbol{p} - \boldsymbol{p}_0\|^2\big) = 0, \qquad (11)$$

where $\|\boldsymbol{p}\|$ is the norm in $\mathbb{C}^n$; the first and second derivatives of the functions $a_0(\boldsymbol{p}), a_1(\boldsymbol{p})$ are expressed in terms of the derivatives of the function $f(z, \boldsymbol{p})$ by formulas (10); the coefficients α_{ij} are determined, according to (5), by the relations

$$\alpha_{00} = \alpha_{11} = \frac{2}{\partial^2 f/\partial z^2}, \qquad \alpha_{10} = -\frac{2}{3}\frac{\partial^3 f/\partial z^3}{(\partial^2 f/\partial z^2)^2}. \qquad (12)$$

Equations (10)–(12) describe the surface $D = 0$ in explicit form via the derivatives of the function $f(z, \boldsymbol{p})$ up to second-order terms. A similar method can be used to obtain relations describing the bifurcation diagram in the neighborhood of the root of arbitrary multiplicity.

The bifurcation diagram of the family of functions $f(z, \boldsymbol{p})$ has a certain physical meaning. Suppose that $f(z, \boldsymbol{p})$ is a polynomial in z resulting from the substitution $z = -\lambda^2$ into the characteristic polynomial of a linear Hamiltonian or nonconservative (circulation) mechanical system. Then the part of the bifurcation diagram characterized by the positive real roots is the boundary of the domain of stability corresponding to the oscillatory form of the loss of stability (flutter) [5 6]. Moreover, relation (11) describes the boundary of the domain of stability locally up to second-order terms.

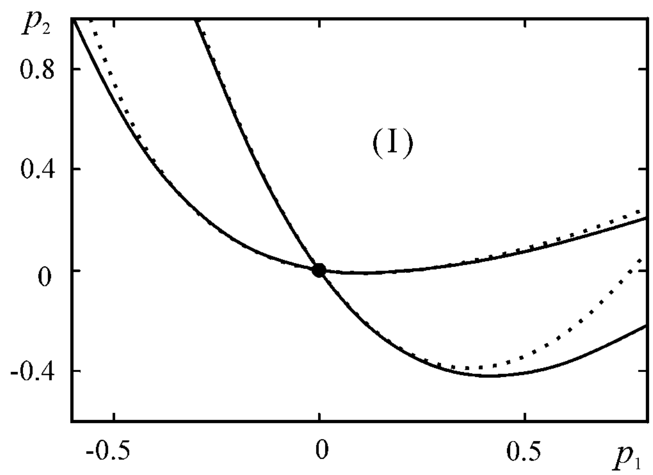

FIG. 1. Bifurcation diagram of the family of polynomials and its approximation

For example, consider the two-parameter polynomial function

$$f(z, \boldsymbol{p}) = z^5 - 13z^4 + \left(62 + \frac{1}{4}p_1\right)z^3 - (134 + p_1^2)z^2 + (129 + p_2)z - 45 + p_1 p_2.$$

For $\boldsymbol{p}_0 = (0,0)$, the polynomial $f(z,\boldsymbol{p}_0)$ has double roots $z=1$, $z=3$ and a simple root $z=5$. Therefore, the bifurcation diagram in the neighborhood of the point $\boldsymbol{p}_0 = (0,0)$ consists of two curves. These curves are given by the equation $D=0$ written at the points $(z_0, p_1^0, p_2^0) = (1,0,0)$ and $(3,0,0)$. Using relations (10)–(12), we can find the equations of the curves $D=0$ up to second-order terms:

$$\begin{aligned} z_0 = 1: &\quad \frac{1}{16}p_1 + \frac{1}{4}p_2 - \frac{15595}{65536}p_1^2 + \frac{2525}{8192}p_1p_2 + \frac{229}{4096}p_2^2 = 0, \\ z_0 = 3: &\quad \frac{27}{8}p_1 + \frac{3}{2}p_2 - \frac{13815}{4096}p_1^2 + \frac{661}{512}p_1p_2 + \frac{49}{256}p_2^2 = 0. \end{aligned} \tag{13}$$

The exact form of the bifurcation diagram and its approximation (13) are shown in Fig. 1 by the solid and the dotted line, respectively. In Fig. 1 the domain (I) defined by the inequalities $D > 0$ corresponds to the presence of only real roots $z(\boldsymbol{p})$. Note that the approximate equations for the bifurcation diagram (13) were calculated from the values of the derivatives of the function $f(z,\boldsymbol{p})$ at the points $(z_0, p_1^0, p_2^0) = (1,0,0)$ and $(3,0,0)$.

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